43-rd Vietnamese Mathematical Olympiad 2005

First Day

1. The real numbers \( x \) and \( y \) satisfy the condition
\[
x - 3\sqrt{x+1} = 3\sqrt{y+2} - y.
\]
Find the maximum and minimum values of \( P = x + y \).

2. Points \( A \) and \( B \) are given on a circle with center \( O \) and radius \( R \) so that \( O \) does not lie on \( AB \). Let \( C \) be a variable point on the circle, distinct from \( A \) and \( B \). Circle \( O_1 \) through \( A \) touches \( BC \) at \( C \), and circle \( O_2 \) through \( B \) touches \( AC \) at \( C \). Circles \( O_1 \) and \( O_2 \) meet at \( C \) and \( D \).

(a) Prove that \( CD \leq R \).

(b) Prove that all lines \( CD \) pass through a fixed point.

3. Let \( A_1A_2 \ldots A_8 \) be a convex octagon with no three diagonals concurrent. We call intersections of its diagonals buttons, and convex quadrilaterals determined by four vertices of the octagon sub-quadrilaterals. Find the smallest \( n \) for which it is possible to color \( n \) buttons so that for all distinct \( i,k \in \{1, \ldots, 8\} \) number \( s(i,k) \) is the same, where \( s(i,k) \) denotes the number of sub-quadrilaterals with two vertices at \( A_i,A_k \) whose diagonals meet at a colored button.

Second Day

4. Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfying the condition
\[
f(f(x-y)) = f(x)f(y) - f(x) + f(y) - xy \quad \text{for all} \ x, y.
\]

5. Find all triples of natural numbers \( (x, y, n) \) such that \( x! + y! = 3^n \).

6. The sequence \( (x_n) \) is defined by \( x_1 = a \in \mathbb{R} \) and
\[
x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n \quad \text{for} \ n = 1, 2, 3, \ldots
\]
Find all \( a \) for which \( (x_n) \) has a finite limit when \( n \rightarrow \infty \) and determine that limit.