42-nd Vietnamese Mathematical Olympiad 2004

1. Solve the system of equations

$$x^3 + x(y-z)^2 = 2, \quad y^3 + y(z-x)^2 = 30, \quad z^3 + z(x-y)^2 = 16.$$

2. Solve the system of equations

$$x^3 + 3xy^2 = -49, \quad x^2 - 8xy + y^2 = 8y - 17x.$$

3. In a triangle ABC, the bisector of $\angle ACB$ cuts the side AB at D. An arbitrary circle o passing through C and D meets the lines BC and AC at M and N (different from C) respectively.

   (a) Prove that there is a circle s touching DM at M and DN at N.

   (b) If circle s intersects the lines BC and CA again at P and Q respectively, prove that the lengths of the segments MP and NQ are constant as o varies.

4. Let ABC be an acute triangle with orthocenter H. Point P distinct from B and C is taken on the shorter arc BC of its circumcircle, and D is the point such that $\overrightarrow{AD} = \overrightarrow{PC}$. Let K be the orthocenter of $\triangle ACD$ and E and F be the orthogonal projections of K on BC and AB. Prove that the line EF bisects the segment HK.

5. The sequence $(x_n)_{n=1}^{\infty}$ is defined by $x_1 = 1$ and

$$x_{n+1} = \frac{(2+\cos 2\alpha)x_n - \cos^2\alpha}{(2-2\cos 2\alpha)x_n + 2 - \cos 2\alpha} \qquad forn \in \mathbb{N},$$

   where α is a given real parameter. Find all values of α for which the sequence (y_n) given by $y_n = \sum_{k=1}^{n} \frac{1}{2x_k+1}$ has a finite limit when $n \to \infty$ and find that limit.

6. Find the minimum and maximum value of the expression

$$P = \frac{x^4 + y^4 + z^4}{(x+y+z)^4}$$

   where x, y, z are positive numbers satisfying $(x+y+z)^3 = 32xyz$.

7. Find all triples (x, y, z) of positive integers such that

$$(x+y)(1+xy) = 2^z.$$

8. Find the least positive integer k with the following property: In each k-element subset of $A = \{1, 2, \ldots, 16\}$ there exist two distinct elements a and b such that $a^2 + b^2$ is a prime number.

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9. Prove that for all integers n, k with $n \geq 2$ and $2n-3 \leq k \leq \frac{n(n-1)}{2}$ there exist n distinct real numbers $a_1, \dots, a_n$ such that among their pairwise sums $a_i + a_j$, $1 \leq i < j \leq n$ there are exactly k different numbers.

10. Let $S(n)$ be the sum of decimal digits of a natural number n. Find the least value of $S(m)$ if m is an integral multiple of 2003.

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