41-st Vietnamese Mathematical Olympiad 2003

First Day - March 13

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(\cot x) = \cos 2x + \sin 2x$ for all $x \in (0,\pi)$. Set $g(x) = f(x)f(1-x)$ for $-1 \le x \le 1$. Find the minimum and maximum values of g on the interval $[-1,1]$.

2. On the plane are given two circles k_1 and k_2 with centers at O_1 and O_2 respectively that are externally tangent at point M. The radius of k_2 is greater than that of k_1. For a point A on k_2 and not on the line O_1O_2, let AB and AC be the two tangents to k_1 (with B and C on k_1). The lines MB and MC cut k_2 at E and F respectively, and the line EF meets the tangent to k_2 drawn at A in a point D. Show that when A assumes all possible positions on k_2, D will lie on a fixed line.

3. For any integer $n > 1$, let S_n denote the number of permutations $(a_1, a_2, \dots, a_n)$ of numbers $1, 2, \dots, n$ such that $1 \le |a_k - k| \le 2$ for $k = 1, 2, \dots, n$. Prove that

$$\frac{3}{4}S_{n-1} < S_n < 2S_{n-1} \quad \text{for all } n > 6.$$

Second Day - March 14

4. Find the greatest positive integer n for which the system

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2$$

has an integer solution $(x, y_1, \dots, y_n)$.

5. Consider the polynomials

$$P(x) = 4x^3 - 2x^2 - 15x + 9 \quad \text{and} \quad Q(x) = 12x^3 + 6x^2 - 7x + 1.$$

   (a) Prove that each of these polynomials has three distinct real zeros.

   (b) If α and β are the greatest zeros of P and Q respectively, show that $\alpha^2 + 3\beta^2 = 4$.

6. Let $\mathscr{F}$ be the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition

$$f(3x) \ge f(f(2x)) + x \quad \text{for all } x > 0.$$

Find the greatest real number α with the property that $f(x) \ge \alpha x$ for all $f \in \mathscr{F}$ and $x > 0$.

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