38-th Vietnamese Mathematical Olympiad 2000

First Day

1. Given a real number $c > 0$ and an initial value x_0 with $0 < x_0 < c$, a sequence (x_n) of real numbers is defined by

$$x_{n+1} = \sqrt{c - \sqrt{c + x_n}} \quad \text{for } n \geq 0.$$

Find all values of c such that for each initial value x_0 in $(0, c)$, the sequence (x_n) is defined for all n and has a finite limit.

2. Two circles Ω_1 and Ω_2 with respective centers O_1, O_2 are given on a plane. Let M_1, M_2 be points on Ω_1, Ω_2 respectively, and let the lines O_1M_1 and O_2M_2 meet at Q. Starting simultaneously from these positions, the points M_1 and M_2 move clockwise on their own circles with the same angular velocity.

   (a) Determine the locus of the midpoint of M_1M_2.

   (b) Prove that the circumcircle of $\triangle M_1QM_2$ passes through a fixed point.

3. Consider the polynomial $P(x) = x^3 + 153x^2 - 111x + 38$.

   (a) Prove that there are at least nine integers a in the interval $[1, 3^{2000}]$ for which $P(a)$ is divisible by 3^{2000}.

   (b) Find the number of integers a in $[1, 3^{2000}]$ with the property from (a).

Second Day

4. For every integer $n \geq 3$ and any given angle α with $0 < \alpha < \pi$, let

$$P_n(x) = x^n \sin\alpha - x \sin n\alpha + \sin(n-1)\alpha.$$

   (a) Prove that there is a unique polynomial of the form $f(x) = x^2 + ax + b$ which divides $P_n(x)$ for every $n \geq 3$.

   (b) Prove that there is no polynomial $g(x) = x + c$ which divides $P_n(x)$ for every $n \geq 3$.

5. Find all integers $n \geq 3$ such that there are n points $A_1, A_2, \ldots, A_n$ in space, with no three on a line and no four on a circle, such that all the circumcircles of the triangles $A_iA_jA_k$ are congruent.

6. Let $P(x)$ be a nonzero polynomial such that, for all real numbers x,

$$P(x^2 - 1) = P(x)P(-x).$$

Determine the maximum possible number of real roots of $P(x)$.

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