45-th All-Ukrainian Mathematical Olympiad 2005

Final Round – March 20–26

Grade 8

First Day

1. Find all pairs of real numbers (x, y) such that
$$\frac{x-2}{y}+\frac{5}{xy}=\frac{4-y}{x}-\frac{|y-2x|}{xy}.$$

2. Let AD be the median of a triangle ABC. Suppose that $\angle ADB = 45^\circ$ and $\angle ACB = 30^\circ$. Compute $\angle BAD$.

3. A 6×6 board is filled out with positive integers. Each move consists of selecting a square larger than 1×1, consisting of entire cells, and increasing all numbers inside the selected square by 1. Is it always possible to perform several moves so as to reach a situation where all numbers on the board are divisible by 3?

Second Day

4. For which real numbers $x > 1$ is there a triangle with side lengths $x^4+x^3+2x^2+x+1$, $2x^3+x^2+2x+1$, and x^4-1?

5. Are there integers a, b, c, d, x, y, z, t such that each of the numbers
$$|ay-bx|,\ |az-cx|,\ |at-dx|,\ |bz-cy|,\ |bt-dy|,\ |ct-dz|$$
equals either 1 or 2005?

6. A convex quadrilateral $ABCD$ with $BC = CD$ and $\angle CBA + \angle DAB > 180^\circ$ is given. Suppose that W and Q are points on the sides BC and DC respectively (distinct from the vertices) such that $AD = QD$ and $WQ \parallel AD$. Also suppose that AQ and BD intersect at a point M that is equidistant from the lines AD and BC. Prove that $\angle BWD = \angle ADW$.

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Grade 9

First Day

1. Draw the locus of points $M(x,y)$ in the Cartesian plane xOy satisfying $(x^2-1)(|y|-1)\geq 0$.

2. Find all pairs of positive integers (m,n) such that $\sqrt{m}+\frac{2005}{\sqrt{n}}=2006$.

3. On the plane are given $n\geq 3$ points, not all on the same line. For any point M on the same plane, $f(M)$ is defined to be the sum of the distances from M to these n points. Suppose that there is a point M_1 such that $f(M_1)\leq f(M)$ for any point M on the plane. Prove that if a point M_2 satisfies $f(M_1)=f(M_2)$, then $M_2\equiv M_1$.

4. Mykolka the numismatist possesses 241 coins of the total amount 360 tughriks (the Mongolian currency), where the value in tughriks of each coin is an integer. Does it necessarily follow that these coins can be divided into three heaps of equal amount?

Second Day

5. Can 1 be written as a sum of 2005 different terms of the form $\frac{1}{3n-1}$, where n is a positive integer?

6. For every positive integer n, prove the inequality
$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{n+2}{n!+(n+1)!+(n+2)!}<\frac{1}{2}.$$

7. Under the rules of the ”Sea battle” game (no two ships may have a common point), can the following sets of rectangular ships be arranged on a 10×10 square board:

   (a) 2 ships 1×4, 4 ships 1×3, 6 ships 1×2, and 8 ships 1×1;

   (b) 2 ships 1×4, 4 ships 1×3, 6 ships 1×2, 6 ships 1×1, and 1 ship 2×2;

   (c) 2 ships 1×4, 4 ships 1×3, 6 ships 1×2, 4 ships 1×1, and 2 ships 2×2?

8. Let AB and CD be two disjoint chords of a circle. A point E, distinct from A and B, is taken on the chord AB. Consider the arc AB not containing C and D. Using a ruler and a compass, construct a point F on this arc such that $PE/EQ=1/2$, where P and Q are the intersection points of AB with the segments FC and FD, respectively.

Grade 10

First Day

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1. Let $0 \le \alpha, \beta, \gamma \le \pi/2$ satisfy the conditions

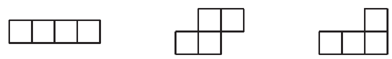
$$\sin\alpha + \sin\beta + \sin\gamma = 1, \quad \sin\alpha\cos 2\alpha + \sin\beta\cos 2\beta + \sin\gamma\cos 2\gamma = -1.$$

   Find all possible values of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

2. Is it possible to divide a 10×10 board into

   (a) 4 figures of type 1 and 21 figures of type 2?

   (b) 4 figures of type 1, 19 figures of type 2, and 2 figures of type 3?



type 1 type 2 type 3

3. An integer $n > 101$ is divisible by 101. Suppose that every divisor d of n with $1 < d < n$ equals the difference of two divisors of n. Prove that n is divisible by 100.

4. Points P and Q do not lie on the diagonal AC of a parallelogram $ABCD$ and satisfy $\angle ABP = \angle ADP$, $\angle CBQ = \angle CDQ$. Prove that $\angle PAQ = \angle PCQ$.

Second Day

5. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ that satisfy

$$f(x)f(y) = f(xy) + 2005\left(\frac{1}{x} + \frac{1}{y} + 2004\right) \quad \text{for all } x, y > 0.$$

6. If a, b, c are positive real numbers, prove the inequality

$$\frac{a^2}{b} + \frac{b^3}{c^2} + \frac{c^4}{a^3} \ge -a + 2b + 2c.$$

7. A point M is taken on the perpendicular bisector of the side AC of an acute-angled triangle ABC so that M and B are on the same side of AC. If $\angle BAC = \angle MCB$ and $\angle ABC + \angle MBC = 180^\circ$, find $\angle BAC$.

8. On the plane are marked 2005 points, no three of which are collinear. A line is drawn through any two of the points. Show that the points can be painted in two colors so that for any two points of the same color the number of the drawn lines separating them is even. (Two points are separated by a line if they lie in different open half-planes determined by the line).

Grade 11

First Day

1. Solve in $\mathbb{R}$ the equation $\left|x-\frac{\pi}{6}\right|+\left|x+\frac{\pi}{3}\right|=\arcsin\frac{x^3-x+2}{2}$.

2. The sum of positive real numbers a,b,c equals 1. Prove that
$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1}+\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1}+\sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1}\geq 6.$$

3. In an acute-angled triangle ABC, ω is the circumcircle and O its center, ω_1 the circumcircle of triangle AOC, and OQ the diameter of ω_1. Let M and N be points on the lines AQ and AC respectively such that the quadrilateral $AMBN$ is a parallelogram. Prove that the lines MN and BQ intersect on ω_1.

4. Find all monotone (not necessarily strictly) functions $f:\mathbb{R}_0^+\to\mathbb{R}$ such that
$$f(x+y)-f(x)-f(y)=f(xy+1)-f(xy)-f(1)\quad \text{for all } x,y\geq 0;$$
$$f(3)+3f(1)=3f(2)+f(0).$$

Second Day

5. Find all positive integers n for which $\cos\left(\pi\sqrt{n^2+n}\right)\geq 0$.

6. A polygon on a coordinate grid is built of 2005 dominoes 1×2. What is the smallest number of sides of an even length such a polygon can have?

7. Prove that for any integer $n\geq 2$ there is a set A_n of n distinct positive integers such that for any two distinct elements $i,j\in A_n$, $|i-j|$ divides i^2+j^2.

8. In space are marked 2005 points, no four of which are in the same plane. A plane is drawn through any three points. Show that the points can be painted in two colors so that for any two points of the same color the number of the drawn planes separating them is odd. (Two points are separated by a plane if they lie in different open half-spaces determined by the plane).

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