7-th Taiwanese Mathematical Olympiad 1998

Time: 4.5 hours each day.

Part 1 – April 25

1. Let m,n be positive integers.

   (a) Prove that $(m,n) = 2\sum_{k=0}^{m-1}\left[\frac{kn}{m}\right] + m + n - mn$.

   (b) If $m,n \geq 2$, prove that

$$\left[\frac{n}{m}\right] + \left[\frac{2n}{m}\right] + \cdots + \left[\frac{(m-1)n}{m}\right] = \left[\frac{m}{n}\right] + \left[\frac{2m}{n}\right] + \cdots + \left[\frac{(n-1)m}{n}\right].$$

2. Does there exist a solution (x,y,z,u,v) in integers greater than 1998 to the equation
$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65?$$

3. Let m,n be positive integers, and let $\mathscr{F}$ be a family of m-element subsets of $\{1,2,\ldots,n\}$ satisfying $A \cap B \neq \emptyset$ for all $A,B \in \mathscr{F}$. Determine the maximum possible number of elements in $\mathscr{F}$.

Part 2 – April 26

4. Let I be the incenter of triangle ABC. Lines AI,BI,CI meet the sides of $\triangle ABC$ at D,E,F respectively. Let X,Y,Z be arbitrary points on segments EF,FD,DE, respectively. Prove that
$$d(X,AB) + d(Y,BC) + d(Z,CA) \leq XY + YZ + ZX,$$
where $d(X,l)$ denotes the distance from a point X to a line l.

5. For a natural number n, let $w(n)$ denote the number of (positive) prime divisors of n. Find the smallest positive integer k such that
$$2^{w(n)} \leq k\sqrt[4]{n} \quad \text{for each } n \in \mathbb{N}.$$

6. In a group of $n \geq 4$ persons, every three who know each other have a common signal. Assume that these signals are not repeated and that there are $m \geq 1$ signals in total. For any set of four persons in which there are three having a common signal, the fourth person has a common signal with at most one of them. Show that there three persons who have a common signal, such that the number of persons having no signal with anyone of them does not exceed $\left[n + 3 - \frac{18m}{n}\right]$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com