6-th Taiwanese Mathematical Olympiad 1997

Time: 4.5 hours each day.

Part 1 – April 14

1. Let a be rational and b,c,d be real numbers, and let $f:\mathbb{R}\to[-1,1]$ be a function satisfying
$$f(x+a+b)-f(x+b)=c\left[x+2a+[x]-2[x+a]-[b]\right]+d$$
for each x. Show that f is periodic.

2. Given a line segment AB in the plane, find all possible points C such that in the triangle ABC, the altitude from A and the median from B have the same length.

3. Let $n\geq 3$ be an integer. Suppose that $a_1,a_2,\dots,a_n$ are real numbers such that $k_i=\dfrac{a_{i-1}+a_{i+1}}{a_i}$ is a positive integer for all i. (Here $a_0=a_n$ and $a_{n+1}=a_1$.) Prove that
$$2n\leq k_1+k_2+\dots+k_n\leq 3n.$$

Part 2 – May 11

4. Let $k=2^{2^n}+1$ for some $n\in\mathbb{N}$. Show that k is prime if and only if k divides $3^{\frac{k-1}{2}}+1$.

5. Let $ABCD$ is a tetrahedron. Show that
   (a) If $AB=CD$, $AC=BD$ and $AD=BC$, then the triangles ABC, ABD, ACD, BCD are acute.
   (b) If the triangles ABC,ABD,ACD,BCD have the same area, then $AB=CD$, $AC=BD$, $AD=BC$.

6. Show that every number of the form 2^p3^q, where p,q are nonnegative integers, divides some number of the form $a_{2k}10^{2k}+a_{2k-2}10^{2k-2}+\dots+a_2 10^2+a_0$, where $a_{2i}\in\{1,2,\dots,9\}$.

Part 3 – June 25

7. Find all positive integers k for which there exists a function $f:\mathbb{N}\to\mathbb{Z}$ satisfying $f(1997)=1998$ and, for all a,b,
$$f(ab)=f(a)+f(b)+kf(\gcd(a,b)).$$

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8. Let O be the circumcenter and R be the circumradius of an acute triangle ABC. Let AO meet the circumcircle of OBC again at D, BO meet the circumcircle of OCA again at E, and CO meet the circumcircle of OAB again at F. Show that

$$OD \cdot OE \cdot OF \geq 8R^3.$$

9. For $n \geq k \geq 3$, let $X = \{1,2,\ldots,n\}$ and let $\mathscr{F}_k$ a the family of k-element subsets of X, any two of which have at most $k-2$ elements in common.Show that there exists a subset M_k of X with at least $[\log_2 n]+1$ elements containing no subset in $\mathscr{F}_k$.

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