5-th Taiwanese Mathematical Olympiad 1996

Time: 4.5 hours each day.

First Day

1. Suppose that α, β, γ are real numbers in the interval $(0, \pi/2)$ such that $\alpha + \beta + \gamma = \pi/4$ and $\tan \alpha = \frac{1}{a}$, $\tan \beta = \frac{1}{b}$, $\tan \gamma = \frac{1}{c}$, where a, b, c are positive integers. Please determine the values of a, b, c.

2. Let $0 < a \le 1$ be a real number and let $a \le a_j \le \frac{1}{a}$ for $j = 1, 2, \dots, 1996$. Show that for any nonnegative real numbers λ_j ($j = 1, 2, \dots, 1996$) with $\sum_{j=1}^{1996} \lambda_j = 1$ it holds that
$$\left(\sum_{i=1}^{1996} \lambda_i a_i \right) \left(\sum_{j=1}^{1996} \frac{\lambda_j}{a_j} \right) \le \left(a + \frac{1}{a} \right)^2.$$

3. Let be given points A and B on a circle, and let P be a variable point on that circle. Let point M be determined by P as the point that is either on segment PA with $AM = MP + PB$ or on segment PB with $AP + MP = PB$. Find the locus of points M.

Second Day

4. Show that for any real numbers $a_3, a_4, \dots, a_{85}$, not all the roots of the equation $a_{85}x^{85} + \dots + a_3x^3 + 3x^2 + 2x + 1$ are real.

5. Determine integers $a_1, a_2, \dots, a_{99} = a_0$ satisfying $|a_{k-1} - a_k| \ge 1996$ for all $k = 1, 2, \dots, 99$, such that the number
$$m = \max\{|a_{k-1} - a_k| \mid k = 1, 2, \dots, 99\}$$
is minimum possible, and find the minimum value m^* of m.

6. Let $(q_n)_{n=0}^{\infty}$ be a sequence of integers such that:

   (i) for any $m > n$, $m - n$ divides $q_m - q_n$, and

   (ii) $|q_n| \le n^{10}$ for all $n \ge 0$.

   Prove that there exists a polynomial $Q(x)$ such that $q_n = Q(n)$ for all n.