4-th Taiwanese Mathematical Olympiad 1995

Time: 4.5 hours each day.

First Day – Taipei, April 13, 1995

1. Let $P(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with complex coefficients, where $a_n = 1$. The roots of $P(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $|\alpha_1|, |\alpha_2|, \ldots, |\alpha_j| > 1$ and $|\alpha_{j+1}|, \ldots, |\alpha_n| \le 1$. Prove that
$$\prod_{i=1}^{j} |\alpha_i| \le \sqrt{|a_0|^2 + \cdots + |a_n|^2}\,.$$

2. Given a sequence of eight integers $x_1, x_2, \ldots, x_8$, in a single operation one replaces these numbers with $|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_8 - x_1|$. Find all the eight-term sequences of integers which reduce to a sequence with all the terms equal after finitely many single operations.

3. Suppose that n persons meet in a meeting, and that each of the persons is acquainted to exactly 8 others. Any two acquainted persons have exactly 4 common acquaintances, and any two non-acquainted persons have exactly 2 common acquaintances. Find all possible values of n.

Second Day – Taipei, April 15, 1995

4. Let $m_1, m_2, \ldots, m_n$ be mutually distinct integers. Prove that there exists a polynomial $f(x)$ of degree n with integer coefficients satisfying the following two conditions:

   (i) $f(m_i) = -1$ for all $i = 1, \ldots, n$;

   (ii) $f(x)$ is irreducible.

5. Let P be a point on the circumcircle of a triangle $A_1A_2A_3$, and let H be the orthocenter of the triangle. The feet B_1, B_2, B_3 of the perpendiculars from P to A_2A_3, A_3A_1, A_1A_2 lie on a line. Prove that this line bisects the segment PH.

6. Let a, b, c, d be integers such that $(a, b) = (c, d) = 1$ and $ad - bc = k > 0$. Prove that there are exactly k pairs (x_1, x_2) of rational numbers with $0 \le x_1, x_2 < 1$ for which both $ax_1 + bx_2$ and $cx_1 + dx_2$ are integers.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com