1-st Taiwanese Mathematical Olympiad 1992

Time: 4.5 hours each day.

First Day – May 3, 1992

1. Let A, B be two points on a given circle, and M be the midpoint of one of the arcs AB. Point C is the orthogonal projection of B onto the tangent l to the circle at A. The tangent at M to the circle meets AC and BC at A' and B' respectively. Prove that if $\angle BAC < \pi/8$, then $S_{ABC} < 2S_{A'B'C}$. (S_X denotes the area of X.)

2. Every positive integer can be represented as a sum of one or more consecutive positive integers. For each n, find the number of such representation of n.

3. If $x_1, x_2, \dots, x_n$ ($n \geq 3$) are positive numbers with $x_1 + x_2 + \cdots + x_n = 1$, prove that
$$x_1^2 x_2 + x_2^2 x_3 + \cdots + x_n^2 x_1 \leq \frac{4}{27}.$$

Second Day – May 5, 1992

4. For a positive integer r, the sequence (a_n) is defined by $a_1 = 1$ and
$$a_{n+1} = \frac{na_n + 2(n+1)^{2r}}{n+2} \qquad \text{for } n \geq 1.$$
Prove that each a_n is a positive integer, and find the n's for which a_n is even.

5. A line through the incenter I of a triangle ABC, perpendicular to AI, intersects AB at P and AC at Q. Prove that the circle tangent to AB at P and to AC at Q is also tangent to the circumcircle of $\triangle ABC$.

6. Find the greatest positive integer A with the following property: For every permutation of the thousand numbers $1001, \dots, 2000$, the sum of some ten consecutive terms is greater than or equal to A.