11-th Taiwanese Mathematical Olympiad 2002

Time: 4.5 hours each day.

Part 1 – Taipei, April 19

1. Find all natural numbers n and nonnegative integers $x_1, \dots, x_n$ such that
$$\sum_{i=1}^{n} x_i^2 = 1 + \frac{4}{4n+1}\left(\sum_{i=1}^{n} x_i\right)^2.$$

2. A lattice point X in the plane is said to be *visible* from the origin O if the line segment OX does not contain any other lattice points. Show that for any positive integer n, there is a square $ABCD$ of area n^2 such that none of the lattice points inside the square is visible from the origin.

3. Suppose x,y,z,a,b,c,d,e,f are real numbers satisfying

   (i) $\max\{a,0\}+\max\{b,0\} < x+ay+bz < 1+\min\{a,0\}+\min\{b,0\}$,

   (ii) $\max\{c,0\}+\max\{d,0\} < cx+y+dz < 1+\min\{c,0\}+\min\{d,0\}$, and

   (iii) $\max\{e,0\}+\max\{f,0\} < ex+fy+z < 1+\min\{e,0\}+\min\{f,0\}$.

   Show that $0 < x,y,z < 1$.

Part 2 – Taipei, May 8

4. Let $0 < x_1,x_2,x_3,x_4 \le \frac{1}{2}$ be real numbers. Prove the inequality
$$\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} \le \frac{x_1^4+x_2^4+x_3^4+x_4^4}{(1-x_1)^4+(1-x_2)^4+(1-x_3)^4+(1-x_4)^4}.$$

5. Suppose that the real numbers $a_1, a_2, \dots, a_{2002}$ satisfy
$$\begin{aligned} \frac{a_1}{2}+\frac{a_2}{3}+\cdots+\frac{a_{2002}}{2003} &= \frac{4}{3};\\ \frac{a_1}{3}+\frac{a_2}{4}+\cdots+\frac{a_{2002}}{2004} &= \frac{4}{5};\\ \cdots\cdots\cdots\cdots &\quad \cdots\\ \frac{a_1}{2003}+\frac{a_2}{2004}+\cdots+\frac{a_{2002}}{4004} &= \frac{4}{4005}. \end{aligned}$$
   Evaluate the sum $\dfrac{a_1}{3}+\dfrac{a_2}{5}+\cdots+\dfrac{a_{2002}}{4005}$.

6. Let A,B,C be fixed points in the plane, and D be a variable point on the circle ABC, distinct from A,B,C. Let I_A,I_B,I_C,I_D be the Simson lines of A,B,C,D with respect to triangles BCD,ACD,ABD,ABC, respectively. Find the locus of the intersection points of the four lines I_A,I_B,I_C,I_D when point D varies.

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