9-th Taiwanese Mathematical Olympiad 2000

Time: 4.5 hours each day.

Part 1 – Taipei, April 7

1. Find all pairs (x,y) of positive integers such that $y^{x^2} = x^{y+2}$.

2. In an acute triangle ABC we have $AC > BC$. Let M be the midpoint of AB, AP and BQ altitudes of the triangle and H the orthocenter. Lines AB and PQ meet at R. Show that $RH \perp CM$.

3. Consider the set $S = \{1,2,\dots,100\}$ and the family $\mathscr{P} = \{T \subset S \mid |T| = 49\}$. Each $T \in \mathscr{P}$ is labeled by an arbitrary number from S. Prove that there exists a subset M of S with $|M| = 50$ such that for each $x \in M$, $M \setminus \{x\}$ is not labelled by x.

Part 2 – Taipei, April 29

4. Suppose that for some $m,n \in \mathbb{N}$ we have $\varphi(5^m - 1) = 5^n - 1$, where φ denotes the Euler function. Show that $(m,n) > 1$.

5. Let n be a positive integer and $A = \{1,2,\dots,n\}$. A subset of A is said to be *connected* if it consists of one element orseveral consecutive elements. Determine the maximum k for which there exist k distinct subsets of A such that the intersection of any two of them is connected.

6. Define a function $f : \mathbb{N} \to \mathbb{N}_0$ by $f(1) = 0$ and

$$f(n) = \max_j \{ f(j) + f(n-j) + j \} \qquad \text{for all } n \geq 2.$$

Determine $f(2000)$.

www.imo.org.yu

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com