Turkish IMO Team Selection Test 1999

First Day – March 20, 1999

1. Let be given positive integers $m \le n$ and a prime number p. Let

$$\begin{aligned} m &= a_0 + a_1 p + \cdots + a_r p^r, \\ n &= b_0 + b_1 p + \cdots + b_s p^s, \end{aligned}$$

   where $a_r, b_s \neq 0$ and $0 \le a_i, b_j < p$ for all i, j. If $a_i \le b_i$ for all $i = 0, 1, \dots, r$, we write $m \prec_p n$. Prove that $p \nmid \binom{n}{m}$ if and only if $m \prec_p n$.

2. Let L and N be the midpoints of the diagonals AC and BD of a cyclic quadrilateral $ABCD$, respectively. If BD bisects the angle ANC, prove that AC bisects the angle BLD.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the set $\{\frac{f(x)}{x} \mid x \neq 0\}$ is finite, and for all $x \in \mathbb{R}$,
$$f(x - 1 - f(x)) = f(x) - x - 1.$$

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4. Let the area and the perimeter of a cyclic quadrilateral C be A_C and P_C, and let the area and the perimeter of the quadrilateral which is tangent to the circumcircle of C at the vertices of C be A_T and P_T, respectively. Prove that $\frac{A_C}{A_T} \ge \left(\frac{P_C}{P_T}\right)^2$.

5. Each of A, B, C, D, E, and F knows a piece of gossip. They communicate by phone via a central switchboard, which can connect only two of them at a time. During a conversation, each side tells the other everything he or she knows at that point. Find the minimum number of calls needed for everyone to know all six pieces of gossip.

6. Prove that the plane is not a union of the inner regions of finitely many parabolas.