Turkish IMO Team Selection Test 1997

First Day – April 12, 1997

1. In a triangle ABC with a right angle at A, H is the foot of the altitude from A. Prove that the sum of the inradii of the triangles ABC, ABH, and AHC is equal to AH.

2. The sequences (a_n), (b_n) are defined by $a_1 = \alpha$, $b_1 = \beta$,

$$a_{n+1} = \alpha a_n - \beta b_n, \quad \beta_{n+1} = \beta a_n + \alpha b_n \quad \text{for all } n > 0.$$

   How many pairs (α, β) of real numbers are there such that $a_{1997} = b_1$ and $b_{1997} = a_1$?

3. In a soccer league, whenever a player is transfered from a team X with x players to a team Y with y players, the federation is paid $y - x$ billions liras by Y if $y \geq x$, while the federation pays $x - y$ billions liras to X if $x > y$. A player is allowed to change as many teams as he wishes during a season. Suppose that a season started with 18 teams of 20 players each. At the end of the season, 12 of the teams turn out to have again 20 players, while the remaining 6 teams end up with $16, 16, 21, 22, 22, 23$ players, respectively. What is the maximal amount the federation may have won during the season?

Second Day – April 13, 1997

4. A convex $ABCDE$ is inscribed in a unit circle, AE being its diameter. If $AB = a$, $BC = b$, $CD = c$, $DE = d$ and $ab = cd = \frac{1}{4}$, compute $AC + CE$ in terms of a, b, c, d.

5. Show that for each prime $p \geq 7$, there exist a positive integer n and integers x_i, y_i ($i = 1, \ldots, n$), not divisible by p, such that

$$x_i^2 + y_i^2 \equiv x_{i+1}^2 \pmod{p}, \quad \text{where } x_{n+1} = x_1.$$

6. If $x_1, x_2, \ldots, x_n$ are positive real numbers with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find the minimum value of

$$\sum_{i=1}^{n} \frac{x_i^5}{x_1 + x_2 + \cdots + x_n - x_i}.$$

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