Turkish IMO Team Selection Test 1996

First Day – March 23, 1996

1. Let
$$\prod_{n=1}^{1996}(1+nx^{3n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m},$$
where $a_1, \dots, a_m$ are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

2. In a parallelogram $ABCD$ with $\angle A < 90^\circ$, the circle with diameter AC intersects the lines CB and CD again at E and F, and the tangent to this circle at A meets the line BD at P. Prove that the points P, E, F are collinear.

3. If $0 = x_1 < x_2 < \cdots < x_{2n+1} = 1$ are real numbers with $x_{i+1} - x_i \le h$ for $1 \le i \le 2n$, show that
$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) \le \frac{1+h}{2}.$$

Second Day – March 24, 1996

4. The diagonals AC and BD of a convex quadrilateral $ABCD$ with $S_{ABC} = S_{ADC}$ intersect at E. The lines through E parallel to AD, DC, CB, BA meet AB, BC, CD, DA at K, L, M, N, respectively. Compute the ratio
$$\frac{S_{KLMN}}{S_{ABCD}}.$$

5. Find the maximum number of pairwise disjoint sets of the form
$$S_{a,b} = \{n^2 + an + b \mid n \in \mathbb{Z}\}, \quad a, b \in \mathbb{Z}.$$

6. Determine all ordered pairs of positive real numbers (a, b) such that every sequence (x_n) satisfying
$$\lim_{n\to\infty}(ax_{n+1} - bx_n) = 0$$
must have the limit 0 as $n \to \infty$.

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