Turkish IMO Team Selection Test 1995

First Day – April 15, 1995

1. Given real numbers $b \geq a > 0$, find all solutions of the system

$$\begin{aligned} x_1^2 + 2ax_1 + b^2 &= x_2, \\ x_2^2 + 2ax_2 + b^2 &= x_3, \\ &\cdots\cdots\cdots \\ x_n^2 + 2ax_n + b^2 &= x_1. \end{aligned}$$

2. Let n be a positive integer. Find the number of permutations σ of the set $\{1, 2, \dots, n\}$ such that $\sigma(j) \geq j$ holds for exactly two values of j.

3. Let D be a point on the small arc AC of the circumcircle of an equilateral triangle ABC, different from A and C. Let E and F be the projections of D onto BC and AC respectively. Find the locus of the intersection point of EF and OD, where O is the center of ABC.

Second Day – April 16, 1995

4. In a convex quadrilateral $ABCD$ it is given that $\angle CAB = 40^\circ$, $\angle CAD = 30^\circ$, $\angle DBA = 75^\circ$ and $\angle DBC = 25^\circ$. Find $\angle BDC$.

5. Let $n \in \mathbb{N}$ be given. Prove that the following two conditions are equivalent:

   (i) $n \mid a^n - a$ for any positive integer a;

   (ii) For any prime divisor p of n, $p^2 \nmid n$ and $p - 1 \mid n - 1$.

6. The sequence $\{x_n\}$ of real numbers is defined by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = x_n + \sqrt[3]{x_n} \ \text{ for } n \geq 1.$$

   Show that there exist real numbers a, b such that $\lim_{n\to\infty} \frac{x_n}{an^b} = 1$.

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