Turkish IMO Team Selection Test 2008

First Day

1. Given a triangle ABC, assume that $\angle B > \angle C$. The internal and external bisectors of $\angle A$ intersect BC at D and E respectively. P is a variable point on EA such that A belongs to the segment EP. The line DP intersects AC at M and ME intersects AD at Q. Prove that the lines PQ have a common point independent of P.

2. A graph has 30 vertices, 105 edges and 4822 unordered pairs of edges whose endpoints are discjoint. Find the maximal possible difference of degrees of two vertices in this graph.

3. The equation $x^3 - ax^2 + bx - c = 0$ has three (not necessarily different) positive real roots. Find the minimal possible value of
$$\frac{1+a+b+c}{3+2a+b} - \frac{c}{b}.$$

Second Day

4. The sequence (x_n) is defined as: $x_1 = a$, $x_2 = b$, and for all positive integers n, $x_{n+2} = 2008x_{n+1} - x_n$. Prove that there are some positive integers a and b such that $1 + 2006x_{n+1}x_n$ is a perfect square for all positive integers n.

5. Let D be a point on the edge BC of a triangle ABC such that
$$AD = \frac{BD^2}{AB+AD} = \frac{CD^2}{AC+AD}.$$
Let E be a point for which D belongs to the segment AE and
$$CD = \frac{DE^2}{CD+CE}.$$
Prove that $AE = AB + AC$.

6. There are n voters and m candidates. Every voter makes his/her arrangement of all candidates (there is one person in every place $1, 2, \dots, m$) and votes for the first k people in his/her list. k candidates with most votes are elected (we call them *winners*). Consider the candidate a and two elections R and R'. The election R' is (R,a)-*good* if for every voter v, the candidates that are in worse position on v's list than a in the election R are also in the worse position than a in R'. We say that a positive integer k is monotone if for every election R and every winner a for R, a is also a winner for all (R,a)-good elections. Prove that k is monotone if and only if
$$k > \frac{m(n-1)}{n}.$$

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