7-th Turkish Mathematical Olympiad 1999/2000

Second Round

First Day – December 3, 1999

1. Find the number of ordered quadruples (x,y,z,w) of integers with $0 \le x,y,z,w \le 36$ such that
$$x^2+y^2 \equiv z^3+w^3 \pmod{37}.$$

2. Let be given a circle with center O. The two tangents from a point S outside the circle touch the circle at P and Q. Let X be an interior point of the smaller arc PB. The line SO intersects the circle at A and B, and meets the lines QX and PX at points C and D, respectively. Prove that
$$\frac{1}{AC}+\frac{1}{AD}=\frac{2}{AB}.$$

3. Let n and p be positive integers. Prove that the number of functions f from the set $\{1,2,\dots,n\}$ to $\{-p,-p+1,\dots,p\}$, satisfying $|f(i)-f(j)| \le p$ for all i,j, is equal to $(p+1)^{n+1}-p^{n+1}$.

Second Day – December 4, 1999

4. Determine all sequences (a_m) of real numbers such that $\sum_{n=1}^{2000} a_n = 1999$ and
$$\frac{1}{2} < a_n < 1 \quad \text{and} \quad a_n = a_{n-1}(2-a_{n-1}) \text{ for all } n>1.$$

5. In an acute triangle $A_1A_2A_3$ whose circumradius is R, Y_1,Y_2,Y_3 are the feet of the altitudes from A_1,A_2,A_3, respectively. Let $A_1Y_1=h_1$, $A_2Y_2=h_2$, and $A_3Y_3=h_3$. If t_1,t_2,t_3 are the lengths of the tangents from A_1,A_2,A_3 to the circumcircle of $\triangle Y_1Y_2Y_3$ respectively, prove that
$$\sum_{i=1}^{3}\frac{t_i^2}{h_i} \le \frac{3}{2}R.$$

6. We wish to find the sum of 40 given numebrs utilizing 40 processors. Initially, we have the number 0 on the screen of each processor. Each processor adds the number on its screen with a number entered directly or transferred from another processor in a unit time. Whenever a number is transferred from a processor to another, the former processor resets. Find the least time needed to find the desired sum.

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com