4-th Turkish Mathematical Olympiad 1996/97

Second Round

First Day – December 6, 1996

1. Let $(A_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ be sequences of positive integers. Assume that for each positive integer x, there is a unique positive integer N and a unique N-tuple $(x_1, x_2, \dots, x_N)$ such that

$$0 \le x_k \le a_k \text{ for } k = 1, 2, \dots, N, \quad x_N \ne 0, \quad \text{and} \quad x = \sum_{k=1}^{N} x_k A_k.$$

   (a) Prove that $A_k = 1$ for some k;

   (b) Prove that $A_k = A_j$ if and only if $k = j$;

   (c) Prove that if $A_k \le A_j$, then $A_k \mid A_j$.

2. Let $ABCD$ be a square of side length 2, and let M and N be points on the sides AB and CD respectively. The lines CM and BN meet at P, while the lines AN and DM meet at Q. Prove that $PQ \ge 1$.

3. Let n integers on the real axis be colored. Determine for which positive integers k there exists a family $\mathscr{K}$ of closed intervals with the following properties:

   (i) The union of the intervals in $\mathscr{K}$ contains all the colored points;

   (ii) Any two distinct intervals in $\mathscr{K}$ are disjoint;

   (iii) For each interval I in $\mathscr{K}$ we have $a_I = kb_I$, where a_I denotes the number of integers in I, and b_I the number of colored integers in I.

Second Day – December 7, 1996

4. A circle is tangent to the sides AD, DC, CB of a convex quadrilateral $ABCD$ at K, L, M, respectively. A line l, passing through L and parallel to AD, meets KM at N and KC at P. Prove that $PL = PN$.

5. Prove that $\prod_{k=0}^{n-1}(2^n - 2^k)$ is divisible by $n!$ for all positive integers n.

6. Show that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x+y) > f(x)(1 + yf(x)) \quad \text{for all } x, y > 0.$$

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