2-nd Turkish Mathematical Olympiad 1994/95

Second Round

First Day – December 23, 1994

1. For $n \in \mathbb{N}$, let a_n denote the closest integer to $\sqrt{n}$. Evaluate $\sum_{n=1}^{\infty} \frac{1}{a_n^3}$.

2. Let $ABCD$ be a cyclic quadrilateral with $\angle BAD < 90^\circ$ and $\angle BCA = \angle DCA$. Point E is taken on segment DA such that $BD = 2DE$. The line through E parallel to CD intersects the diagonal AC at F. Prove that $AC \cdot BD = 2AB \cdot FC$.

3. Let n blue lines, no two of which are parallel and no three concurrent, be drawn on a plane. An intersection of two blue lines is colored blue. Through any two blue points that have not already been joined by a blue line, a red line is drawn. An intersection of two red points is colored red, and an intersection of a red line and a blue line is colored purple. What is the maximum possible number of purple points?

Second Day – December 24, 1994

4. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. For each $u \in \mathbb{R}^+$, we denote $g(u) = \inf\{f(t) + u/t \mid t > 0\}$. Prove that:
   (a) If $x \leq g(xy)$, then $x \leq 2f(2y)$;
   (b) If $x \leq f(y)$, then $x \leq g(xy)$.

5. Find the set of all ordered pairs (s,t) of positive integers such that $t^2 + 1 = s(s+t)$.

6. The incircle of a triangle ABC touches BC at D and AC at E. Let K be the point on CB with $CK = 3BD$, and L be the point on CA with $AE = CL$. Lines AK and BL meet at P. If Q is the midpoint of BC, I the incenter, and G the centroid of $\triangle ABC$, show that:
   (a) IQ and AK are parallel,
   (b) the triangles AIG and QPG have equal area.

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