14-th Turkish Mathematical Olympiad 2006/07

Second Round

First Day – December 16, 2006

1. Points P and Q on side AB of a convex quadrilateral $ABCD$ are given such that $AP = BQ$. The circumcircles of triangles APD and BQD meet again at K and those of APC and BQC meet again at L. Show that the points D, C, K, L lie on a circle.

2. There are 2006 students and 14 teachers in a school. Each student knows at least one teacher (knowing is a symmetric relation). Suppose that, for each pair of a student and a teacher who know each other, the ratio of the number of the students whom the teacher knows to that of the teachers whom the student knows is at least t. Find the maximum possible value of t.

3. Find all positive integers n for which all coefficients of polynomial $P(x)$ are divisible by 7, where
$$P(x) = (x^2+x+1)^n - (x^2+1)^n - (x+1)^n - (x^2+x)^n + x^{2n} + x^n + 1.$$

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4. Positive real numbers $x_1, \dots, x_n$ have both the sum and the sum of their squares equal to t. Prove that
$$\sum_{i \neq j} \frac{x_i}{x_j} \geq \frac{(n-1)^2 t}{t-1}.$$

5. The incircle of a triangle ABC touches the sides BC, CA, AB respectively at N_A, N_B, N_C, and H_A, H_B, H_C are the respective feet of the altitudes from A, B, C. The incenters of triangles AH_BH_C, BH_CH_A and CH_AH_B are I_A, I_B, I_C, respectively. Prove that the hexagon $I_AN_BI_CN_AI_BN_C$ is regular.

6. Find all triangles whose area, side lengths and angle measures (in degrees) are all rational numbers.