12-th Turkish Mathematical Olympiad 2004/05

Second Round

First Day – December 11, 2004

1. In a triangle ABC with $\angle B > \angle C$, the altitude, the angle bisector, and the median from A intersect BC at H, L, and D, respectively. Show that $\angle HAL = \angle DAL$ if and only if $\angle BAC = 90^\circ$.

2. Two-way flights are operated between 80 cities in such a way that each city is connected to at least 7 otyher cities by a direct flight and any two cities are connected by a finite sequence of flights. Find the smallest k such that for any such arrangement of flights it is possible to travel from any city to any other city by a sequence of at most k flights.

3. (a) For each $k = 1, 2, 3$ find an integer n such that $n^2 - k$ has exactly 10 positive divisors.

   (b) Show that the number of positive divisors of $n^2 - 4$ is not 10 for any integer n.

Second Day – December 12, 2004

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition

$$f(n) - f(n + f(m)) = m \quad \text{for all } m, n \in \mathbb{Z}.$$

5. The excircle of a triangle ABC corresponding to A touches the lines BC, CA, AB at A_1, B_1, C_1, respectively. The excircle corresponding to B touches BC, CA, AB at A_2, B_2, C_2, and the excircle corresponding to C touches BC, CA, AB at A_3, B_3, C_3, respectively. Find the maximum possible value of the ratio of the sum of the perimeters of $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, and $\triangle A_3B_3C_3$ to the circumradius of $\triangle ABC$.

6. Define $K(n, 0) = \emptyset$ and, for all nonnegative integers m and n,

$$K(n, m+1) = \{k \in \mathbb{N} \mid k \le n \text{ and } K(k, m) \cap K(n-k, m) = \emptyset\}.$$

   Find the number of elements of $K(2004, 2004)$.

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