10-th Turkish Mathematical Olympiad 2002/03

First Day

1. There are $n \geq 2$ cars participating in a rally. The cars leave the start line at different times and finish the rally at different taimes. It turned out that during the entire rally each car took over any other car at most once, the number of cars taken over by each car is different, and each car is taken over by the same number of cars. Find all possible values of n.

2. Let $ABCD$ be a convex quadrilateral and let K, L, M, and N be the midpoints of AB, BC, CD, and DA, respectively. Prove that

$$\sqrt[3]{S_{AKN}} + \sqrt[3]{S_{BKL}} + \sqrt[3]{S_{CLM}} + \sqrt[3]{S_{DMN}} \leq 2\sqrt[3]{S_{ABCD}}.$$

3. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2)$$

for all $x_1, x_2 \in \mathbb{R}$ and all $t \in (0,1)$. Prove that

$$\sum_{k=1}^{2003} f(a_{k+1}a_k \geq \sum_{k=1}^{2003} f(a_k)a_{k+1}$$

for all real numbers a_1, a_2, $\dots$, a_{2004} such that $a_1 \geq a_2 \geq \cdots \geq a_{2003}$ and $a_{2004} = a_1$.

Second Day

4. Find all positive integers a, n, p, k such that $k > 1$, and

$$a^{2n+1} + a^{n+1} + 1 = p^k.$$

5. A circle tangent to the sides AB and BC of $\triangle ABC$ is also tangent to its circumcircle at the point T. If I is the incenter of $\triangle ABC$, prove that $\angle ATI = \angle CTI$.

6. A labeling of all squares of a $m \times n$ chessboard by 0s and 1s is called *fair* if the total number of 0s and the total number of 1s are equal. A real number $a \in [0, \frac{1}{2}]$ is called *beautiful* if there are positive integers m and n and a fair labeling of the $m \times n$ chessboard such that the percentage of 1s in each row and each column is between a and $100 - a$. Find the largest beautiful number.

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com