Swiss IMO Team Selection Tests 1998

First Test
May 7

1. A function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ has the following properties:

   (i) $f(x) - f(y) = f(x) f\left(\frac{1}{y}\right) - f(y) f\left(\frac{1}{x}\right)$ for all $x, y \neq 0$;

   (ii) f takes the value $\frac{1}{2}$ at least once.

   Determine $f(-1)$.

2. Find all nonnegative integer solutions (x,y,z) of the equation
$$\frac{1}{x+2} + \frac{1}{y+2} = \frac{1}{2} + \frac{1}{z+2}.$$

3. Given positive numbers a,b,c, find the minimum of the function
$$f(x) = \sqrt{a^2+x^2} + \sqrt{(b-x)^2+c^2}.$$

4. Find all numbers n for which it is possible to cut a square into n smaller squares.

5. Points A and B are chosen on a circle k. Let AP and BQ be segments of the same length tangent to k, drawn on different sides of line AB. Prove that the line AB bisects the segment PQ.

Second Test
May 23

1. Find all prime numbers p for which p^2+11 has exactly six positive divisors.

2. Consider an $n \times n$ matrix whose entry at the intersection of the i-th row and the j-th column equals $i+j-1$. What is the largest possible value of the product of n entries of the matrix, no two of which are in the same row or column?

3. Let $\triangle ABC$ be an equilateral triangle and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points X, Y, Z respectively. Prove that
$$XY \cdot YZ \cdot ZX \geq XB \cdot YC \cdot ZA.$$

4. If x and y are positive numbers, prove the inequality
$$\frac{x}{x^4+y^2} + \frac{y}{x^2+y^4} \leq \frac{1}{xy}.$$

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5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies for all $x \in \mathbb{R}$

   (i) $|f(x)| \leq 1$, and

   (ii) $f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$.

   Prove that f is a periodic function.

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