Swiss IMO Team Selection Tests 2004

First Test

May 15

1. Let S be the set of all n-tuples $(X_1, \ldots, X_n)$ of subsets of the set $\{1, 2, \ldots, 1000\}$, not necessarily different and not necessarily nonempty. For $a = (X_1, \ldots, X_n)$ denote by $E(a)$ the number of elements of $X_1 \cup \cdots \cup X_n$. Find an explicit formula for the sum
$$\sum_{a \in S} E(a).$$

2. Find the largest natural number n for which $4^{995} + 4^{1500} + 4^n$ is a square.

3. Let ABC be an isosceles triangle with $AC = BC$, whose incenter is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Second Test

May 16

1. Let a, b, and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1.$$
When does equality hold?

2. A brick has the shape of a cube of size 2 with one corner unit cube removed. Given a cube of size 2^n divided into unit cubes from which an arbitrary unit cube is removed, show that the remaining figure can be build using the described bricks.

3. Find all finite sequences $(x_0, x_1, \ldots, x_n)$ such that for every k, $0 \leq k \leq n$, x_k equals the number of times k appears in the sequence.

Third Test

June 12

1. The real numbers a, b, c, d satisfy the equations:
$$a = \sqrt{45 - \sqrt{21 - a}}, \quad b = \sqrt{45 + \sqrt{21 - b}},$$
$$c = \sqrt{45 - \sqrt{21 + c}}, \quad d = \sqrt{45 + \sqrt{21 + d}}.$$
Prove that $abcd = 2004$.

2. Let m be a fixed integer greater than 1. The sequence $x_0, x_1, x_2, \ldots$ is defined as follows:
$$x_i = \begin{cases} 2^i, & \text{if } 0 \le i \le m-1; \\ \sum_{j=1}^{m} x_{i-j}, & \text{if } i \ge m. \end{cases}$$
Find the greatest k for which the sequence contains k consecutive terms divisible by m.

3. Let $A_1, A_2, \ldots, A_n$ be different subsets of an n-element set X. Show that there exists $x \in X$ such that the sets
$$A_1 \setminus \{x\},\ A_2 \setminus \{x\},\ \ldots,\ A_n \setminus \{x\}$$
are all different.

Fourth Test

June 13

1. In an acute-angled triangle ABC the altitudes AU, BV, CW intersect at H. Points X, Y, Z, different from H, are taken on segments AU, BV, and CW, respectively.
   (a) Prove that if X, Y, Z and H lie on a circle, then the sum of the areas of triangles ABZ, AYC, XBC equals the area of ABC.
   (b) Prove the converse of (a).

2. Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real $x \neq y$
$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)}.$$

3. Find all natural numbers which can be written in the form
$$\frac{(a+b+c)^2}{abc}, \quad \text{where } a,b,c \in \mathbb{N}.$$

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