Swiss IMO Team Selection Tests 2003

First Test

May 9

1. Real numbers x,y,a satisfy the equations
$$x+y=x^3+y^3=x^5+y^5=a.$$
Find all possible values of a.

2. In an acute-angled triangle ABC, E and F are the feet of the altitudes from B and C, and G and H are the projections of B and C on EF, respectively. Prove that $HE=FG$.

3. Find the largest real number C_1 and the smallest real number C_2 such that all real numbers a,b,c,d,e satisfy
$$C_1<\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+e}+\frac{e}{e+a}<C_2.$$

4. Find the largest natural number n that divides $a^{25}-a$ for all integers a.

5. There are n pieces on the squares of a 5×9 board, at most one on each square at any time during the game. A move in the game consists of simultaneously moving each piece to a neighboring square by side, under the restriction that a piece having been moved horizontally in the previous move must be moved vertically and vice versa. Find the greatest value of n for which there exists an initial position starting at which the game can be continued until the end of the world.

Second Test

May 24

1. If a,b,c are positive numbers with $a+b+c=2$, prove the inequality
$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\geq\frac{27}{13}$$
and determine when equality holds.

2. Find all polynomials $Q(x)=ax^2+bx+c$ with integer coefficients for which there exist three different prime numbers p_1,p_2,p_3 such that
$$|Q(p_1)|=|Q(p_2)|=|Q(p_3)|=11.$$

3. Let $A_1A_2A_3$ be a triangle and ω_1 be a circle passing through A_1 and A_2. Suppose that there are circles $\omega_2,\dots,\omega_7$ such that:

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(a) ω_k passes through A_k and A_{k+1} for $k = 2, 3, \dots, 7$, where $A_i = A_{i+3}$;

(b) ω_k and ω_{k+1} are externally tangent for $k = 1, 2, \dots, 6$.

Prove that $\omega_1 = \omega_7$.

4. Given integers $0 < a_1 < a_2 < \cdots < a_{101} < 5050$, prove that one can always choose for different numbers a_k, a_l, a_m, a_n such that $5050 \mid a_k + a_l - a_m - a_n$.

5. Find all strictly monotonous functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy $f(f(n)) = 3n$ for all $n \in \mathbb{N}$.

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