Swiss IMO Team Selection Tests 2001

First Test

May 4

1. The 2001×2001 trees in a park form a square grid. What is the largest number of trees that can be cut so that no tree stump can be seen from any other? (Each tree has zero width.)

2. If a, b, and c are the sides of a triangle, prove the inequality
$$\sqrt{a+b-c}+\sqrt{c+a-b}+\sqrt{b+c-a} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$
When does equality occur?

3. In a convex pentagon every diagonal is parallel to one side. Show that the ratios between the lengths of diagonals and the sides parallel to them are equal and find their value.

4. For a natural number $n \geq 2$, consider all representations of n as a sum of its distinct divisors, $n = t_1 + t_2 + \cdots + t_k$, $t_i \mid n$. Two such representations differing only in order of the summands are considered the same (for example, $20 = 10+5+4+1$ and $20 = 5+1+10+4$). Let $a(n)$ be the number of different representations of n in this form. Prove or disprove: There exists M such that $a(n) \leq M$ for all $n \geq 2$.

5. Let $a_1 < a_2 < \cdots < a_n$ be a sequence of natural numbers such that for $i < j$ the decimal representation of a_i does not occur as the leftmost digits of the decimal representation of a_j. (For example, 137 and 13729 cannot both occur in the sequence.) Prove that
$$\sum_{i=1}^{n} \frac{1}{a_i} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9}.$$

Second Test

May 19

1. A function $f : [0,1] \to \mathbb{R}$ has the following properties:
   (a) $f(x) \geq 0$ for $0 < x < 1$;
   (b) $f(1) = 1$;
   (c) $f(x+y) \geq f(x) + f(y)$ whenever $x, y, x+y \in [0,1]$.

   Prove that $f(x) \leq 2x$ for all $x \in [0,1]$.

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2. Let ABC be an acute-angled triangle with circumcenter O. The circle $\mathscr{S}$ through A,B, and O intersects AC and BC again at points P and Q respectively. Prove that $CO \perp PQ$.

3. Find two smallest natural numbers n for which each of the fractions

$$\frac{68}{n+70}, \frac{69}{n+71}, \frac{70}{n+72}, \dots, \frac{133}{n+135}$$

is irreducible.

4. In Geneva there are 16 secret agents, each of whom is watching one or more other agents. It is known that if agent A is watching agent B, then B is not watching A. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc, the last is watching the first. Show that any 11 agents can also be so ordered.

5. Prove that every 1000-element subset M of the set $\{0,1,\dots,2001\}$ contains either a power of two or two distinct numbers whose sum is a number of two.