Swedish Mathematical Competition 1997

Final Round

November 22, 1997

1. Let AC be a diameter of a circle and AB be tangent to the circle. The segment BC intersects the circle again at D. Show that if $AC = 1$, $AB = a$, and $CD = b$, then
$$\frac{1}{a^2 + \frac{1}{2}} < \frac{b}{a} < \frac{1}{a^2}.$$

2. Let D be the point on side AC of a triangle ABC such that BD bisects $\angle B$, and E be the point on side AB such that $3\angle ACE = 2\angle BCE$. Suppose that BD and CE intersect at a point P with $ED = DC = CP$. Determine the angles of the triangle.

3. Let A and B be integers with an odd sum. Show that every integer can be written in the form $x^2 - y^2 + Ax + By$, where x, y are integers.

4. Players A and B play the following game. Each of them throws a dice, and if the outcomes are x and y respectively, a list of all two digit numbers $10a + b$ with $a, b \in \{1, \dots, 6\}$ and $10a + b \leq 10x + y$ is created. Then the players alternately reduce the list by replacing a pair of numbers in the list by their absolute difference, until only one number remains. If the remaining number is of the same parity as the outcome of A's throw, then A is proclaimed the winner. What is the probability that A wins the game?

5. Let $s(m)$ denote the sum of (decimal) digits of a positive integer m. Prove that for every integer $n > 1$ not equal to 10 there is a unique integer $f(n) \geq 2$ such that
$$s(k) + s(f(n) - k) = n$$
for all integers k with $0 < k < f(n)$.

6. Assume that a set M of real numbers is the union of finitely many disjoint intervals with the total length greater than 1. Prove that M contains a pair of distinct numbers whose difference is an integer.

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