Swedish Mathematical Competition 1996

Final Round

November 22, 1996

1. Through an arbitrary point inside a triangle, lines parallel to the sides of the triangle are drawn, dividing the triangle into three triangles with areas T_1, T_2, T_3 and three parallelograms. If T is the area of the original triangle, prove that

$$T = \left(\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}\right)^2.$$

2. In the country of Postonia, one wants to have only two values of stamps. These values should be integers greater than 1 with the difference 2, and should have the proparty that one can combine the stamps for any postage which is greater than or equal to the sum of these two values. What values can be chosen?

3. For every positive integer n, we define the function p_n for $x \geq 1$ by

$$p_n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right).$$

   Prove that $p_n(x) \geq 1$ and that $p_{mn}(x) = p_m(p_n(x))$.

4. The angles at A, B, C, D, E of a pentagon $ABCDE$ inscribed in a circle form an increasing sequence. Show that the angle at C is greater than $\pi/2$, and that this lower bound cannot be improved.

5. Let $n \geq 1$. Prove that it is possible to select some of the integers $1, 2, \ldots, 2^n$ so that for each $p = 0, 1, \ldots, n-1$ the sum of the p-th powers of the selected numbers is equal to the sum of the p-th powers of the remaining numbers.

6. A rectangle is tiled with rectangles of size 6×1. Prove that one of its side lengths is divisible by 6.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com