Swedish Mathematical Competition 1988

Final Round

November 19, 1988

1. Let $a > b > c$ be sides of a triangle and h_a, h_b, h_c be the corresponding altitudes. Prove that $a + h_a > b + h_b > c + h_c$.

2. Six ducklings swim on the surface of a pond, which is in the shape of a circle with radius 5m. Show that at every moment, two of the ducklings swim on the distance of at most 5m from each other.

3. Show that if $x_1 + x_2 + x_3 = 0$ for real numbers x_1, x_2, x_3, then $x_1x_2 + x_2x_3 + x_3x_1 \leq 0$.

   Find all $n \geq 4$ for which $x_1 + x_2 + \cdots + x_n = 0$ implies $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \leq 0$.

4. A polynomial $P(x)$ of degree 3 has three distinct real roots. Find the number of real roots of the equation

$$P'(x)^2 - 2P(x)P''(x) = 0.$$

5. Show that there exists a constant $\alpha > 1$ such that, for any positive integers m and n,

$$\frac{m}{n} < \sqrt{7} \quad \text{implies that} \quad 7 - \frac{m^2}{n^2} \geq \frac{\alpha}{n^2}.$$

6. The sequence (a_n) is defined by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n^2 + \frac{1}{a_n}}$ for $n \geq 1$. Prove that there exists α such that

$$\frac{1}{2} \leq \frac{a_n}{n^\alpha} \leq 2 \quad \text{for } n \geq 1.$$

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