Swedish Mathematical Competition 1984

Final Round

November 17, 1984

1. Let A and B be two points inside a circle C. Show that there exists a circle that contains A and B and lies completely inside C.

2. The squares in a 3×7 grid are colored either blue or yellow. Consider all $m\times n$ rectangles in this grid, where $m\in\{2,3\}$, $n\in\{2,\dots,7\}$. Prove that at least one of these rectangles has all four corner squares the same color.

3. Prove that if a,b are positive numbers, then
$$\left(\frac{a+1}{b+1}\right)^{b+1}\geq\left(\frac{a}{b}\right)^{b}.$$

4. Find all positive integers p and q such that all the roots of the polynomial
$$(x^2-px+q)(x^2-qx+p)$$
are positive integers.

5. Solve in natural numbers a,b,c the system
$$\begin{aligned} a^3-b^3-c^3 &= 3abc, \\ a^2 &= 2(a+b+c). \end{aligned}$$

6. Assume $a_1,a_2,\dots,a_{14}$ are positive integers such that
$$\sum_{i=1}^{14}3^{a_i}=6558.$$
Prove that the numbers $a_1,\dots,a_{14}$ consist of the numbers $1,\dots,7$, each taken twice.

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