46-th Swedish Mathematical Competition 2006

Final Round

Luleå, November 25, 2006

1. If positive integers a and b have 99 and 101 different positive divisors respectively (including 1 and the number itself), can the product ab have exactly 150 positive divisors?

2. In a triangle ABC, point P is the incenter and A', B', C' its orthogonal projections on BC, CA, AB, respectively. Show that $\angle B'A'C'$ is acute.

3. A cubic polynomial f with a positive leading coefficient has three different positive zeros. Show that $f'(a) + f'(b) + f'(c) > 0$.

4. Saskia and her sisters have been given a large number of pearls. The pearls are white, black and red, not necessarily the same number of each color. Each white pearl is worth 5 Ducates, each black one is worth 7, and each red one is worth 12. The total worth of the pearls is 2107 Ducates. Saskia and her sisters split the pearls so that each of them gets the same number of pearls and the same total worth, but the color distribution may vary among the sisters. Interestingly enough, the total worth in Ducates that each of the sisters holds equals the total number of pearls split between the sisters. Saskia is particularly fond of the red pearls, and therefore makes sure that she has as many of those as possible. How many pearls of each color has Saskia?

5. In each square of an $m \times n$ rectangular board there is a nought or a cross. Let $f(m,n)$ be the number of such arrangements that contain a row or a column consisting of noughts only. Let $g(m,n)$ be the number of arrangements that contain a row consisting of noughts only, or a column consisting of crosses only. Which of the numbers $f(m,n)$ and $g(m,n)$ is larger?

6. Determine all positive integers a, b, c satisfying $a^{(b^c)} = (b^a)^c$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com