28-th Spanish Mathematical Olympiad 1992

Second Round

First Part

1. A natural number N is divisible by 83 and N^2 has exactly 63 divisors. Find the smallest N with these properties.

2. Given two circles of radii r and r' exterior to each other, construct a line parallel to a given line and intersecting the two circles in chords with the sum of lengths l.

3. Prove that if a,b,c,d are nonnegative integers satisfying

$$(a+b)^2+2a+b=(c+d)^2+2c+d,$$

then $a=c$ and $b=d$.

Show that the same is true if a,b,c,d satisfy $(a+b)^2+3a+b=(c+d)^2+3c+d$, but show that there exist a,b,c,d with $a\neq c$ and $b\neq d$ satisfying $(a+b)^2+4a+b=(c+d)^2+4c+d$.

Second Part

4. Prove that the arithmetic progression $3,7,11,15,\dots$ contains infinitely many prime numbers.

5. Given a triangle ABC, show how to construct the point P such that

$$\angle PAB=\angle PBC=\angle PCA.$$

Express this angle in terms of $\angle A,\angle B,\angle C$ using trigonometric functions.

6. For a positive integer n, let $S(n)$ be the set of complex numbers $z=x+iy$ ($x,y\in\mathbb{R}$) with $|z|=1$ satisfying

$$(x+iy)^n+(x-iy)^n=2x^n.$$

   (a) Determine $S(n)$ for $n=2,3,4$.

   (b) Find an upper bound (depending on n) of the number of elements of $S(n)$ for $n>5$.

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