25-th Spanish Mathematical Olympiad 1989

Second Round

Madrid, February

First Part

1. An exam at a university consists of one question randomly selected from the n possible questions. A student knows only one question, but he can take the exam n times. Express as a function of n the probability p_n that the student will pass the exam. Does p_n increase or decrease as n increases? Compute $\lim_{n\to\infty} p_n$. What is the largest lower bound of the probabilities p_n?

2. Points A', B', C' on the respective sides BC, CA, AB of triangle ABC satisfy $\frac{AC'}{AB} = \frac{BA'}{BC} = \frac{CB'}{CA} = k$. The lines AA', BB', CC' form a triangle $A_1B_1C_1$ (possibly degenerate). Given k and the area S of $\triangle ABC$, compute the area of $\triangle A_1B_1C_1$.

3. Prove that $\dfrac{1}{10\sqrt{2}} < \dfrac{1\cdot 3\cdot 5\cdots 99}{2\cdot 4\cdot 6\cdots 100} < \dfrac{1}{10}$.

Second Part

4. Show that the number 1989, as well as each of its powers 1989^n ($n \in \mathbb{N}$), can be expressed as a sum of two positive squares in at least two ways.

5. Consider the set D of all complex numbers of the form $a + b\sqrt{-13}$ with $a, b \in \mathbb{Z}$. The number $14 = 14 + 0\sqrt{-13}$ can be written as a product of two elements of D: $14 = 2\cdot 7$. Find all possible ways to express 14 as a product of two elements of D.

6. Prove that among any seven real numbers there exist two, a and b, such that
$$\sqrt{3}|a-b| < |1+ab|.$$
Give an example of six real numbers not having this property.

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