24-th Spanish Mathematical Olympiad 1988

Second Round

Madrid, February 1988

First Part

1. A sequence of integers $(x_n)_{n=1}^{\infty}$ satisfies $x_1 = 1$ and $x_n < x_{n+1} \le 2n$ for all n. Show that for every positive integer k there exist indices r, s such that $x_r - x_s = k$.

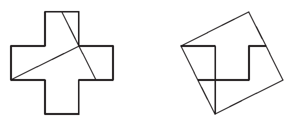
2. We choose $n > 3$ points on a circle and number them 1 to n in some order. We say that two non-adjacent points A and B are *related* if, in one of the arcs AB, all the points are marked with numbers less than those at A, B. Show that the number of pairs of related points is exactly $n - 3$.

3. Prove that if one of the numbers $25x + 3y$, $3x + 7y$ (where $x, y \in \mathbb{Z}$) is a multiple of 41, then so is the other.

Second Part

4. The Fibonacci sequence is given by $a_1 = 1$, $a_2 = 2$ and $a_{n+1} = a_n + a_{n-1}$ for $n > 1$. Express a_{2n} in terms of only a_{n-1}, a_n, a_{n+1}.

5. A well-known puzzle asks for a partition of a cross into four parts which are to be reassembled into a square. One solution is exhibited on the picture.



   Show that there are infinitely many solutions. (Some solutions split the cross into four equal parts!)

6. For all integral values of parameter t, find all integral solutions (x, y) of the equation
$$y^2 = x^4 - 22x^3 + 43x^2 - 858x + t^2 + 10452(t + 39).$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com