21-st Spanish Mathematical Olympiad 1985

Second Round

February 1985

First Part

1. Let $f : P \to P$ be a bijective map from a plane P to itself such that:

   (i) $f(r)$ is a line for every line r,

   (ii) $f(r)$ is parallel to r for every line r.

   What possible transformations can f be?

2. Determine if there exists a subset E of $\mathbb{Z} \times \mathbb{Z}$ with the properties:

   (i) E is closed under addition;

   (ii) E contains $(0,0)$;

   (iii) For every $(a,b) \neq (0,0)$, E contains exactly one of (a,b) and $-(a,b)$.

   Remark: We define $(a,b)+(a',b') = (a+a', b+b')$ and $-(a,b) = (-a,-b)$.

3. Solve the equation $\tan^2 2x + 2\tan 2x \tan 3x = 1$

4. Prove that for each positive integer k there exists a triple (a,b,c) of positive integers such that $abc = k(a+b+c)$. In all such cases prove that $a^3+b^3+c^3$ is not a prime.

Second Part

5. Find the equation of the circle in the complex plane determined by the roots of the equation $z^3 + (-1+i)z^2 + (1-i)z + i = 0$.

6. Let OX and OY be non-collinear rays. Through a point A on OX, draw two lines r_1 and r_2 that are antiparallel with respect to $\angle XOY$. Let r_1 cut OY at M and r_2 cut OY at N. (Thus, $\angle OAM = \angle ONA$). The bisectors of $\angle AMY$ and $\angle ANY$ meet at P. Determine the location of P.

7. Find the values of p for which the equation $x^5 - px - 1 = 0$ has two roots r and s which are the roots of equation $x^2 - ax + b = 0$ for some integers a, b.

8. A square matrix is *sum-magic* if the sum of all elements in each row, column and major diagonal is constant. Similarly, a square matrix is *product-magic* if the product of all elements in each row, column and major diagonal is constant. Determine if there exist 3×3 matrices of real numbers which are both *sum-magic* and *product-magic*.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com
Typed in LaTeX by Ercole Suppa