46-th Spanish Mathematical Olympiad 2010

Valladolid, March 26–27, 2010

First Part

1. A *pucelana* sequence is an increasing sequence of 16 consecutive odd numbers whose sum is a perfect cube. How many pucelana sequences are there with 3-digit numbers only?

2. Let N_0 and Z be the set of all non-negative integers and the set of all integers, respectively. Let $f : N_0 \to Z$ be a function defined as
$$f(n) = -f(\lfloor \frac{n}{3} \rfloor) - 3\{\frac{n}{3}\}$$
where $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x and $\{x\} = x - \lfloor x \rfloor$. Find the smallest integer n with $f(n) = 2010$

3. Let $ABCD$ be a convex quadrilateral. AB and CD meet at P, with $\angle APD = 60^\circ$. Let E, F, G, and H be the midpoints of AB, BC, CD, and DA, respectively. Find the greatest positive real number k for which
$$EG + 3HF \geq kd + (1-k)s$$
where s is the semiperimeter of the quadrilateral $ABCD$ and d is the sum of the lengths of its diagonals. When does the equality hold?

Second Part

4. Let a, b, and c be positive real numbers. Prove that
$$\frac{a+b+3c}{3a+3b+2c} + \frac{a+3b+c}{3a+2b+3c} + \frac{3a+b+c}{2a+3b+3c} \geq \frac{15}{8}.$$

5. In a triangle ABC, let P be a point on the bisector of $\angle BAC$ and let A', B' and C' be points on lines BC, CA and AB respectively such that PA' is perpendicular to BC, $PB' \perp AC$, and $PC' \perp AB$. Prove that PA' and $B'C'$ intersect on the median AM, where M is the midpoint of BC.

6. Let p be a prime number and A an infinite subset of the natural numbers. Let $f_A(n)$ be the number of different solutions of $x_1 + x_2 + \ldots + x_p = n$, with $x_1, x_2, \ldots, x_p \in A$. Does there exist a number N for which $f_A(n)$ is constant for all $n < N$?

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