45-th Spanish Mathematical Olympiad 2009

Sant Feliu de Guíxols, March 27–28, 2009

First Part

1. Find all finite sequences consisting of n consecutive natural numbers a_1, a_2, $\cdots$, a_n, with $n \geq 3$, such that $a_1 + a_2 + \cdots + a_n = 2009$.

2. Let ABC be an acute triangle, I the center of its inscribed circle, r its radius, and R the radius of the circumcircle of $\triangle ABC$. We draw altitude $AD = h_a$, with D on side BC. Prove that
$$DI^2 = (2R - h_a)(h_a - 2r).$$

3. Some edges of a regular polyhedron are painted red. A *good* coloring is one in which, for each vertex, there is a non-red edge originating at that vertex. A coloring is *completely good* if it is good and no face is completely surrounded by red edges. For which regular polyhedra is the maximum number of edges that can be painted in a good coloring the same as in a completely good coloring?

Second Part

4. Find all pairs of integers (x, y) satisfying
$$x^2 - y^4 = 2009.$$

5. Let a, b, and c be positive real numbers such that $abc = 1$. Prove that
$$(\frac{a}{1+ab})^2 + (\frac{b}{1+bc})^2 + (\frac{c}{1+ca})^2 \geq \frac{3}{4}.$$

6. Given two points A and B inside the circle with center O and radius r, assume that A and B are symmetric with respect to O. We consider a variable point P on the circumference and we draw the chord PP', perpendicular to AP. Let C be the symmetric point to B with respect to PP'. Find the locus of points $Q = AC \cap PP'$, as P varies.

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The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX and translation by Ander Lamaison
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