Slovenian Team Selection Tests 1998

First Test

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f\left((x-y)^2\right) = f(x)^2 - 2xf(y) + y^2 \quad \text{for all } x, y \in \mathbb{R}.$$

2. A semicircle with center O and diameter AB is given. Point M on the extension of AB is taken so that $AM > BM$. A line through M intersects the semicircle at C and D so that $CM < DM$. The circumcircles of triangles AOD and OBC meet again at point K. Prove that OK and KM are perpendicular.

3. (a) Alenka has two jars, each with 6 marbles labeled with numbers 1 through 6. She draws one marble from each jar at random. Denote by p_n the probability that the sum of the labels of the two drawn marbles is n. Compute p_n for each $n \in \mathbb{N}$.

   (b) Barbara has two jars, each with 6 marbles which are labeled with unknown numbers. The sets of labels in the two jars may differ and two marbles in the same jar can have the same label. If she draws one marble from each jar at random, the probability that the sum of the labels of the drawn marbles is n equals the probability p_n in Alenka's case. Determine the labels of the marbles. Find all solutions.

Second Test

1. Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

2. On a line p which does not meet a circle $\mathcal{K}$ with center O, point P is taken such that $OP \perp p$. Let $X \neq P$ be an arbitrary point on p. The tangents from X to $\mathcal{K}$ touch it at A and B. Denote by C and D the orthogonal projections of P on AX and BX respectively.

   (a) Prove that the intersection point Y of AB and OP is independent of the location of X.

   (b) Lines CD and OP meet at Z. Prove that Z is the midpoint of PY.

3. Let $a_0 = 1998$ and $a_{n+1} = \dfrac{a_n^2}{a_n + 1}$ for each nonnegative integer n. Prove that $[a_n] = 1994 - n$ for $0 \leq n \leq 1000$.

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