Slovenian Team Selection Tests 1997

First Test

1. Circles $\mathcal{K}_1$ and $\mathcal{K}_2$ are externally tangent to each other at A and are internally tangent to a circle $\mathcal{K}$ at A_1 and A_2 respectively. The common tangent to $\mathcal{K}_1$ and $\mathcal{K}_2$ at A meets $\mathcal{K}$ at point P. Line PA_1 meets $\mathcal{K}_1$ again at B_1 and PA_2 meets $\mathcal{K}_2$ again at B_2. Show that B_1B_2 is a common tangent of $\mathcal{K}_1$ and $\mathcal{K}_2$.

2. Find all polynomials p with real coefficients such that for all real x
$$xp(x)p(1-x)+x^3+100 \geq 0.$$

3. Let $A_1, A_2, \dots, A_n$ be $n \geq 2$ distinct points on a circle. Find the number of colorings of these points with $p \geq 2$ colors such that every two adjacent points receive different colors.

Second Test

1. Let P be a point in the interior of an equilateral triangle ABC. The lines AP, BP, CP meet the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Prove that
$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A.$$

2. The lattice points of an $n \times n$ grid are to be colored red or blue in such a way that each unit square has exactly two blue vertices. How many different colorings are there?

3. Let p be a prime number and a be an integer. Prove that if $2^p + 3^p = a^n$ for some integer n, then $n = 1$.

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