Slovenian Team Selection Tests 2005

First Test

February 2005

1. The diagonals of a convex quadrilateral $ABCD$ intersect at M. The bisector of $\angle ACD$ intersects the ray BA at K. Prove that if $MA \cdot MC + MA \cdot CD = MB \cdot MD$, then $\angle BKC = \angle BDC$.

2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}+$ such that for any $x, y > 0$,
$$x^2 \left(f(x) + f(y) \right) = (x+y) f\left(f(x)y \right).$$

3. Find all pairs (m,n) of positive integers such that both $m^2 - 4n$ and $n^2 - 4m$ are perfect squares.

Second Test

May 2005

1. Find the number of sequences of 2005 terms with the following properties:
   (i) No three consecutive terms of the sequence are equal;
   (ii) Every term equals either 1 or -1;
   (iii) The sum of all terms of the sequence is at least 666.

2. Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extension points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

3. Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality
$$3\sqrt[3]{\frac{1}{abc} + 6(a+b+c)} \le \frac{\sqrt[3]{3}}{abc}.$$

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