# Slovenian National Mathematical Olympiad 1998

## Final Round

### 1-st Grade

1. Find all integers $x$ and $y$ which satisfy the equation $xy = 20 - 3x + y$.
2. A four-digit number has the property that the units digit equals the tens digit increased by 1, the hundreds digit equals twice the tens digit, and the thousands digit is at least twice the units. Determine this four-digit number, knowing that it is twice a prime number.
3. A point $E$ on side $CD$ of a rectangle $ABCD$ is such that $\triangle DBE$ is isosceles and $\triangle ABE$ is right-angled. Find the ratio between the side lengths of the rectangle.
4. In the lower-left $3 \times 3$ square of an $8 \times 8$ chessboard there are nine pawns. Every pawn can jump horizontally or vertically over a neighboring pawn to the cell across it if that cell is free. Is it possible to arrange the nine pawns in the upper-left $3 \times 3$ square of the chessboard using finitely many such moves?

### 2-nd Grade

1. Find all positive integers $n$ that are equal to the sum of digits of $n^2$.
2. Find all pairs $(p,q)$ of real numbers such that $p+q = 1998$ and the solutions of the equation $x^2 + px + q = 0$ are integers.
3. A point $A$ is outside a circle $\mathscr{K}$ with center $O$. Line $AO$ intersects the circle at $B$ and $C$, and a tangent through $A$ touches the circle in $D$. Let $E$ be an arbitrary point on the line $BD$ such that $D$ lies between $B$ and $E$. The circumcircle of the triangle $DCE$ meets line $AO$ at $C$ and $F$ and line $AD$ at $D$ and $G$. Prove that the lines $BD$ and $FG$ are parallel.
4. Two players play the following game starting with one pile of at least two stones. A player in turn chooses one of the piles and divides it into two or three nonempty piles. The player who cannot make a legal move loses the game. Which player has a winning strategy?

### 3-rd Grade

1. Show that for any integter $a$, the number $\dfrac{a^5}{5} + \dfrac{a^3}{3} + \dfrac{7a}{15}$ is an integer.
2. Find all polynomials $p$ with real coefficients such that for all real $x$

$$(x-8)p(2x) = 8(x-1)p(x).$$



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3. A rectangle $ABCD$ with $AB > AD$ is given. The circle with center $B$ and radius $AB$ intersects the line $CD$ at $E$ and $F$.

   (a) Prove that the circumcircle of triangle $EBF$ is tangent to the circle with diameter $AD$. Denote the tangency point by $G$.

   (b) Prove that the points $D, G$, and $B$ are collinear.

4. Alf was attending an eight-year elementary school on Melmac. At the end of each schoolyear he showed the certificate to his father. If he was promoted, his father gave him the number of cats equal to Alf's age times the number of the grade he passed. During the elementary education Alf failed one grade and had to repeat it. When he finished elementary education he found out that the total number of cats he had received was divisible by 1998. Which grade did Alf fail?

**4-th Grade**

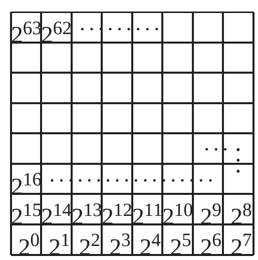
1. Let $n$ be a positive integer. If number 1998 is written in base $n$, a three-digit number with the sum of digits equal to 24 is obtained. Find all possible values of $n$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) + xf(1-x) = x^2 + 1 \quad \text{for all } x \in \mathbb{R}.$$

3. In a right-angled triangle $ABC$ with the hypotenuse $BC$, $D$ is the foot of the altitude from $A$. The line through the incenters of the triangles $ABD$ and $ADC$ intersects the legs of $\triangle ABC$ at $E$ and $F$. Prove that $A$ is the circumcenter of triangle $DEF$.

4. On every square of a chessboard there are as many grains as shown on the picture. Starting from an arbitrary square, a knight starts a journey over the chessboard. After every move it eats up all the grains from the square it arrived to, but when it leaves, the same number of grains is put back on the square. After some time the knight returns to its initial square. Prove that the total number of grains the knight has eaten up during the journey is divisible by 3.



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