Slovenian National Mathematical Olympiad 1997

Final Round

Ljubljana, May 17–18, 1997

1-st Grade

1. Let k be a positive integer. Prove that:

   (a) If $k = m + 2mn + n$ for some positive integers m, n, then $2k + 1$ is composite.

   (b) If $2k + 1$ is composite, then there exist positive integers m, n such that $k = m + 2mn + n$.

2. Let a be an integer and p a prime number that divides both $5a - 1$ and $a - 10$. Show that p also divides $a - 3$.

3. Let MN be a chord of a circle with diameter AB, and let A' and B' be the orthogonal projections of A and B onto MN. Prove that $MA' = B'N$.

4. Janez wants to make an $m \times n$ grid (consisting of unit squares) using equal elements of the form $\llcorner$, where each leg of an element has the unit length. No two elements can overlap. For which values of m and n can Janez do the task?

2-nd Grade

1. Suppose that $a^2 + b^2 + (a+b)^2 = c^2 + d^2 + (c+d)^2$ holds for some real numbers a, b, c, d. Prove that $a^4 + b^4 + (a+b)^4 = c^4 + d^4 + (c+d)^4$.

2. Points M, N, P, Q are taken on the sides AB, BC, CD, DA respectively of a square $ABCD$ such that $AM = BN = CP = DQ = \frac{1}{n}AB$. Find the ratio of the area of the square determined by the lines MN, NP, PQ, QM to the ratio of $ABCD$.

3. Let C and D be different points on the semicircle with diameter AB. The lines AC and BD intersect at E, and the lines AD and BC intersect at F. Prove that the midpoints X, Y, Z of the segments AB, CD, EF respectively are collinear.

4. Prove that among any 1001 numbers taken from the numbers $1, 2, \dots, 1997$ there exist two with the difference 4.

3-rd Grade

1. Suppose that m,n are integers greater than 1 such that $m+n-1$ divides m^2+n^2-1. Prove that $m+n-1$ cannot be a prime number.

2. Determine all positive integers n for which there exists a polynomial $p(x)$ of degree n with integer coefficients such that it takes the value n in n distinct integer points and takes the value 0 at point 0.

3. In a convex quadrilateral $ABCD$ we have $\angle ADB = \angle ACD$ and $AC = CD = DB$. If the diagonals AC and BD intersect at X, prove that $\frac{CX}{BX} - \frac{AX}{DX} = 1$.

4. In an enterprise, no two employees have jobs of the same difficulty and no two of them take the same salary. Every employee gave the following two claims:

   (i) Less than 12 employees have a more difficult work;

   (ii) At least 30 employees take a higher salary.

   Assuming that an employee either always lies or always tells the truth, find how many employees are there in the enterprise.

4-th Grade

1. Marko chose two prime numbers a and b with the same number of digits and wrote them down one after another, thus obtaining a number c. When he decreased c by the product of a and b, he got the result 154. Determine the number c.

2. The Fibonacci sequence f_n is defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \in \mathbb{N}$.

   (a) Show that f_{1005} is divisible by 10.

   (b) Show that f_{1005} is not divisible by 100.

3. Two disjoint circles k_1 and k_2 with centers O_1 and O_2 respectively lie on the same side of a line p and touch the line at A_1 and A_2 respectively. The segment O_1O_2 intersects k_1 at B_1 and k_2 at B_2. Prove that $A_1B_1 \perp A_2B_2$.

4. The expression $*3^5 * 3^4 * 3^3 * 3^2 * 3 * 1$ is given. Ana and Branka alternately change the signs $*$ by $+$ or $-$ (one in turn). Can Branka, who plays second, do this so as to obtain an expression whose value is divisible by 7?

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