24-th All-Russian Mathematical Olympiad 1998

Final Round

Grade 9

First Day

1. Each of the rays $y = x$ and $y = 2x$ ($x \geq 0$) cuts off an arc from a given parabola $y = x^2 + px + q$. Prove that the projection of the first arc to the x-axis is shorter by 1 than that of the second arc. *(The jury)*

2. A convex polygon is partitioned into parallelograms. A vertex of the polygon is called *good* if it belongs to exactly one parallelogram. Prove that there are more than two good vertices. *(M. Smurov)*

3. Let $S(x)$ denote the sum of decimal digits of x. Do there exist natural numbers a, b, c such that
$$S(a+b) < 5, \quad S(a+c) < 5, \quad S(b+c) < 5, \quad S(a+b+c) > 50?$$
(S. Volchenkov, L. Mednikov)

4. A maze is an 8×8 board with some adjacent squares separated by walls, so that any two squares can be connected by a path not meeting any wall. Given a command LEFT, RIGHT, UP or DOWN, a pawn makes a step in the corresponding direction unless it encounters a wall or an edge of the chessboard. God writes a program consisting of a finite sequence of commands and gives it to the Devil, who then constructs a maze and places the pawn on one of the squares. Can God write a program which guarantees that the pawn will visit every square despite the Devil's efforts? *(V. Ufnarovsky, A. Shapovalov)*

Second Day

5. We are given five watches which can be winded forward. What is the smallest sum of winding intervals which allows us to set them to the same time, no matter how they were set initially? *(O. Podlipsky)*

6. Let BL be an internal bisector and BM be a median of a triangle ABC with $AB > BC$. The line through M parallel to AB intersects BL at D, and the line through L parallel to BC intersects BM at E. Prove that the lines ED and BL are perpendicular. *(M. Sonkin)*

7. A jeweller makes a chain consisting of $N > 3$ numbered links. A querulous customer then asks him to change the order of the links, in such a way that the number of links the jeweller must open is maximized. What is this maximum number? *(A. Shapovalov)*

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8. Two positive integers are written on the board. The following operation is iterated: if the numbers a and b are written, the smaller number is replaced with $\frac{ab}{|a-b|}$. Eventually the two numbers become equal. Prove that they are positive integers again. *(I. Izmestyev)*

Grade 10

First Day

1. Two lines parallel to the x-axis cut the graph of $y = ax^3 + bx^2 + cx + d$ in points A, D, E and B, C, F, respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.
(I. Izmestyev)

2. Two polygons are given on the plane. Assume that the distance between any two vertices of the same polygon is at most 1, and that the distance between any two vertices of different polygons is at least $1/\sqrt{2}$. Prove that these two polygons have no common interior points. *(V. Dolnikov)*

3. Let be given a non-isosceles triangle ABC. The tangent from the foot of the bisector of $\angle A$ to the incircle of $\triangle ABC$, other than the line BC, meets the incircle at a point K_a. Points K_b and K_c are analogously defined. Prove that the lines connecting K_a, K_b, K_c with the midpoints of BC, CA, AB, respectively, have a common point on the incircle.
(I. Sharygin)

4. Let k be a positive integer. Some of the $2k$-element subsets of a given set are marked. Suppose that for any subset of cardinality less than or equal to $(k+1)^2$ all the marked subsets contained in it (if any) have a common element. Show that all the marked subsets have a common element.
(V. Dolnikov)

Second Day

5. Initially the numbers 19 and 98 are written on a board. Every minute, each of the two numbers is either squared or increased by 1. Is it possible to obtain two equal numbers at some time? *(Ye. Malinikova)*

6. A binary operation $*$ on real numbers has the property that $(a*b)*c = a+b+c$ for all a, b, c. Prove that $a*b = a+b$. *(B. Frenkin)*

7. A convex n-gon, $n > 3$, is given, no four vertices lying on a circle. A circle is said to be *circumscribed* if it passes through three vertices and contains all the others. A circumscribed circle is *boundary* if it passes through three consecutive vertices of the polygon, and *inner* if it passes through three pairwiose non-consecutive vertices. Prove that the number of boundary circles is two more than the number of inner circles. *(O. Musin)*

8. Each square of a $(2^n-1)\times(2^n-1)$ board contains either 1 or -1. Such an arrangement is called *successful* if each number is the product of its neighbors (two squares are neighboring if they share a side). Find the number of successful arrangements. *(D. Lyubshin)*

Grade 11

First Day

1. *Problem 1 for Grade 10.*

2. The incircle of a triangle ABC touches BC, CA, AB at A_1, B_1, C_1, respectively. Let A_2, B_2, C_2 be the midpoints of the arcs BAC, CBA, ACB of the circumcircle, respectively. Prove that the lines AA_1, BB_1, CC_1 are concurrent. *(M. Sonkin)*

3. A set $\mathscr{S}$ of translates of an equilateral triangle is given in the plane, and any two have nonempty intersection. Prove that there exist three points such that every triangle in $\mathscr{S}$ contains one of these points.
(V. Dolnikov, R. Karasev)

4. There are 1998 cities in Russia, and each of them is connected to three other cities by two-way flights. It is possible to travel between any two cities. The KGB plans to close 200 cities, no two joined by a single flight. Show that this can be done so that one can travel between any two of the remaining cities not passing through a closed city.
(D. Karpov, R. Karasev)

Second Day

5. A sequence of distinct circles $\omega_1, \omega_2, \dots$ is inscribed in the parabola $y=x^2$ so that ω_n and ω_{n+1} are tangent for all n. If ω_1 has diameter 1 and touches the parabola at $(0,0)$, find the diameter of ω_{1998}. *(M. Yevdokimov)*

6. Are there 1998 different positive integers, the product of any two being divisible by the square of their difference? *(G. Galperin)*

7. A tetrahedron $ABCD$ has all edges of length less than 100, and contains two nonintersecting spheres of diameter 1. Prove that it contains a sphere of diameter 1.01. *(R. Karasev)*

8. A figure Φ composed of unit squares has the following property: if the squares of an $m\times n$ rectangle (m,n are fixed) are filled with numbers whose sum is positive, the figure Φ can be placed within the rectangle (possibly after being rotated) so that the sum of the covered numbers is also positive. Prove that a number of such figures can be put on the $m\times n$ rectangle so that each square is covered by the same number of figures.
(A. Belov)