23-rd All-Russian Mathematical Olympiad 1997

Final Round

Grade 9

First Day

1. Let $P(x)$ be a quadratic polynomail with nonnegative coefficients. Prove that for all real numbers x, y,
$$P(xy)^2 \leq P(x^2)P(y^2).$$
(Ye. Malinnikova)

2. A convex polygon M maps to itself by a rotation for the angle of 90°. Show that there are two circles, whose radii are in ratio equal to $\sqrt{2}$, one of which contains M and the other is contained in M. *(A. Hrabrov)*

3. The lateral surface of a rectangular parallelepiped with base $a \times b$ and altitude c (a, b, c are natural numbers) is to be covered by rectangles whose sides are parallel to the edges of the parallelepiped, without overlapping, where each rectangle consists of an even number of unit squares. A rectangle may be folded at an edge of the parallelepiped. Prove that if c is even, then the number of possible coverings is also even.
(D. Karpov, S. Rukshin, D. Fon der Flaass)

4. A test of the Council of Sages is performed in the following way: The king arranges the sages in a line and puts a white or black hat on everyone's head. Every sage can see the hats of all sages that are in front of him, but can't see the hat of anybody behind himself, nor his own hat. Then each of the sages, one by one, guess the color of his own hat. The king punishes all sages who make a wrong guess.

   Before the test, the sages met and devised a way to reduce the number of those who will be punished to a minimum. How many of them can avoid the punishment?

Second Day

5. Do there exist real numbers b and c such that each of the equations $x^2 + bx + c = 0$ and $2x^2 + (b+1)x + (c+1) = 0$ has two integer roots?
(N. Agakhanov)

6. There are 33 pupils in a class. Each of the pupils was asked about the number of pupils in the class with the same given name and the number of those with the same surname. It turned out that each of the numbers from 0 to 10 occured as an answer. Prove that in this class there are two pupils with the same given name and surname. *(A. Shapovalov)*

The IMO Compendium Group,
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7. The incircle of a triangle ABC touches the sides AB, BC, and CA at points M, N, K, respectively. The line through A parallel to NK intersects MN at D. The line through A and parallel to MN intersects NK at E. Prove that the line DE contains two midpoints of sides of the triangle ABC. *(M. Sonkin)*

8. Numbers $1, 2, \dots, 100$ are written in the cells of a 10×10 board, so that the sum of any two adjacent numbers does not exceed S. Find the least possible value of S. (Two numbers are called adjacent if they are written in cells that have a common side.) *(D. Hramtsov)*

Grade 10

First Day

1. Solve in integers the equation
$$(x^2 - y^2)^2 = 1 + 16y.$$
(M. Sonkin)

2. An $n \times n$ square has been folded into a cylinder. Some of its cells are colored black. Prove that there exist two parallel lines - rows, columns or diagonals - which contain the same number of black cells.
(Ye. Poroshenko)

3. Two circles meet at points A and B. A line through A intersects the first circle at C and the second at D. Let M and N be the midpoints of the arcs BC and BD not containing A, and let K be the midpoint of the segment CD. Prove that the angle $\angle MKN$ is right. (It can be assumed that C and D lie on different sides of A.)
(D. Teryoshin)

4. A polygon can be cut into 100 rectangles, but not into 99. Prove that this polygon cannot be cut into 100 triangles. *(A. Shapovalov)*

Second Day

5. Are there two quadratic trinomials with integer coefficients $ax^2 + bx + c$ and $(a+1)x^2 + (b+1)x + (c+1)$, each of which has two integer zeroes?
(N. Agakhanov)

6. A circle with center O is inscribed in a triangle ABC and meets sides AC, AB, BC at points K, M, N, respectively. The median BB_1 of the triangle intersects MN at D. Prove that point O lies on line DK.
(M. Sonkin)

7. Find all triples (m, n, l) of positive integers such that
$$m + n = \gcd(m,n)^2, \quad m + l = \gcd(m,l)^2, \quad n + l = \gcd(n,l)^2.$$

(S. Tokarev)

8. On a strip of cells, infinite in both directions, the cells are numbered by integers. Several stones have been put on the cells of the strip (possibly more than one on a cell). The following moves are permitted:

   (A) Remove one stone from each of the cells $n-1$ and n and put one stone on the cell $n+1$;

   (B) Remove two stones from the cell n and put one on each of the cells $n+1$, $n-2$.

   Prove that, no matter how we perform the moves, we 'll end up in a finite time with a position in which no further moves can be made. Moreover, prove that this final position does not depend on the sequence of moves.

   (D. Fon der Flaass)

Grade 11

First Day

1. *Problem 1 for Grade 10.*

2. A test of the Council of Sages is performed in the following way: The king arranges the sages in a line and puts a white, blue, or red hat on everyone's head. Every sage can see the hats of all sages that are in front of him, but can't see the hat of anybody behind himself, nor his own hat. Then each of the sages, one by one, guess the color of his own hat. The king punishes all sages who make a wrong guess.

   Before the test, the sages met and devised a way to reduce the number of those who will be punished to a minimum. How many of them can avoid the punishment?

   (K. Knop)

3. *Problem 3 for Grade 10.*

4. A cube $n \times n \times n$ is composed from unit cubes. Consider a polygonal line that consists of segments connecting the centers of two adjacent cubes (i.e. sharing a face). We call a face of a unit cube *marked* if the polygonal line passes through it. Prove that the edges of the unit cubes can be painted in two colors in such a way that every marked face has an odd number, and every non-marked face has an even number of edges of each color.

   (M. Smurov)

5. Consider all possible trinomials of the form $x^2 + px + q$, where p, q are integers with $1 \leq p, q \leq 1997$. Which trinomials among them are more numbered: those having integer zeroes, or those having no real roots?

   (M. Yevdokimov)

6. On the plane are given a polygon, a line l, and a point P on l in a general position (i.e. the lines containing sides of the polygon meet l in different points which are all different from P). Let us mark those vertices of the polygon with the property that the lines, containing sides of the polygon at these vertices, meet l on different sides of P. Prove that P lies inside the polygon if and only if each of the half-planes determined by l contains an odd number of marked vertices.

   (O. Musin)

7. The inscribed sphere of a tetrahedron meets one of its faces at its incenter, a second face at its orthocenter, and a third face at its centroid. Show that this tetrahedron is regular.

   (N. Agakhanov)

8. In an $m \times n$ rectangular grid, where m and n are odd positive integers, 1×2 dominoes are initially placed so as to exactly cover all except one of the corner unit squares. It is permitted to slide a domino towards the empty square, thus exposing enother square. Show that by a sequence of such moves, we can move the empty square to any corner of the grid.

   (A. Shapovalov)