# 22-nd All-Russian Mathematical Olympiad 1996

## Final Round – Ryazan', April 19–20

### Grade 9

*First Day*

1. What numbers are more numbered among the integers from 1 to 1000000: those that can be written as a sum of a square and a positive cube, or those that cannot be? *(A. Golovanov)*

2. The centers $O_1, O_2, O_3$ of three nonintersecting congruent circles are situated at the vertices of a triangle. From each of these points one draws tangents to the other two given circles. Suppose that these tangents determine a convex hexagon. The sides of this hexagon are alternately colored red and blue. Prove that the sum of the lengths of the red sides equals the sum of the lengths of the blue sides. *(D. Tereshin)*

3. Let $x, y, p, n, k$ be natural numbers such that $x^n + y^n = p^k$. Prove that if $n > 1$ is an odd number and $p$ an odd prime, then $n$ is a power of $p$. *(A. Kovalji, V. Senderov)*

4. In the duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Show that one can find two committees having at least four common members. *(A. Skopenkov)*

*Second Day*

5. Prove that the arithmetic progression with the first term 1 and the common difference 729 contains infinitely many powers of 10. *(L. Kuptsov)*

6. In an isosceles triangle $ABC$ ($AC = BC$), $O$ is the circumcenter, $I$ the incenter, and $D$ the point on $BC$ such that $OD$ and $BI$ are perpendicular. Prove that the lines $ID$ and $AC$ are parallel. *(M. Sonkin)*

7. Two piles of coins lie on a table. It is known that the total weights of the two piles are equal, and for any natural number $k$ not exceeding the number of coins in either pile, the sum of the weights of the $k$ heaviest coins in the first pile does not exceed that of the second pile. Prove that for every $x > 0$, if if each coin (in either pile) of weight at least $x$ is replaced with a coin of weight $x$, the first pile will not be lighter than the second. *(D. Fon der Flaas)*

8. Can a $5 \times 7$ board be covered by $L$-trominos (i.e. figures formed from a $2 \times 2$ square by removing a corner unit square), not crossing its boundary,



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in several layers, so that each square of the board is covered by the same number of trominos? *(M. Yevdokimov)*

## Grade 10

### *First Day*

1. Points $E$ and $F$ are taken on side $BC$ of a convex quadrilateral $ABCD$ such that $E$ is between $B$ and $F$. Suppose that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$. *(M. Smurov)*

2. There are four counters on a coordinate plane, centered in points with integer coordinates. One can translate a counter by the vector determined by the centers of two of the other counters. Prove that any two preselected counters can be taken to the same point after finitely many moves. *(R. Sadykov)*

3. Find all natural numbers $n$ for which there exist coprime numbers $x$ and $y$ and a natural number $k > 1$ such that $3^n = x^k + y^k$. *(A. Kovalji, V. Senderov)*

4. Prove that if nonzero numbers $a_1, a_2, \dots, a_m$ satisfy for each $k = 0, 1, \dots, n$ ($n < m-1$),
$$a_1 + 2^k a_2 + 3^k a_3 + \cdots + m^k a_m = 0,$$
then the sequence $a_1, \dots, a_m$ contains at least $n+1$ pairs of consecutive terms having opposite signs. *(O. Musin)*

### *Second Day*

5. At the vertices of a cube are written eight distinct natural numbers, and on each edge is written the greatest common divisor of the numbers at its endpoints. Can the sum of the numbers at the vertices be the same as the sum of the numbers at the edges? *(A. Shapovalov)*

6. Three sergeants and several soldiers serve in a platoon. The sergeants take turns on duty. The commander has given the following orders:

   (i) Each day, at least one task must be issued to a soldier.

   (ii) A soldier cannot have more than two tasks nor receive two tasks in the same day.

   (iii) The lists of soldiers receiving tasks for two different days must be different.

   (iv) The first sergeant violating any of these orders will be jailed.

   Can at least one of the sergeants, without consipiring with the others, give tasks according to these rules and avoid being jailed? *(M. Kulikov)*

7. A convex polygon with no two sides parallel is given. For each side we consider the angle the side subtents at the vertex farthest from the side. Prove that the sum of these angles equals $180^\circ$. *(M. Smurov)*

8. Goodnik writes 10 numbers on the board, then Nogoodnik writes 10 more numbers, where all the 20 numbers are positive and distinct. Can Goodnik choose his 10 numbers so that, no matter what Nogoodnik writes, he can form 10 quadratic trinomials of the form $x^2+px+q$, where $p$ and $q$ run through all the written numbers, such that the real roots of these trinomials take exactly 11 values? *(I. Rubanov)*

## Grade 11

*First Day*

1. Can the number obtained by writing the numbers from 1 to $n$ one after another be a palindrom? *(N. Agakhanov)*

2. Several hikers travel at fixed speeds along a straight road. It is known that during some period of time the sum of their pairwise distances is monotonically decreasing. Show that there is a hiker, the sum of whose distances to the other hikers is monotonically decreasing during this period. *(A. Shapovalov)*

3. Show that for $n \geq 5$ a cross-section of a pyramid whose base is a regular $n$-gon cannot be a regular $n+1$-gon. *(N. Agakhanov, D. Tereshin)*

4. *Problem 4 for Grade 10.*

*Second Day*

5. Are there three natural numbers greater than 1 such that the square of each of them, decreased by one, is divisible by each of the remaining numbers? *(A. Golovanov)*

6. In an isosceles triangle $ABC$ ($AB = BC$), $CD$ is the bisector of angle $C$. The line through the circumcenter of $ABC$ perpendicular to $CD$ meets $BC$ at $E$. The line through $E$ parallel to $CD$ meets $AB$ at $F$. Prove that $BE = FD$. *(M. Sonkin)*

7. Does there exist a finite set $M$ of nonzero real numbers, such that for any $n \in \mathbb{N}$ there is a polynomial of degree at least $n$ with coefficients in $M$, whose all roots belong to $M$? *(Ye. Malinnikova)*

8. The numbers from 1 to 100 are written in an unknown order. One may ask about the relative order of any 50 numbers. What is the smallest number of questions needed to find the order of all 100 numbers? *(S. Tokarev)*