15-th All-Russian Mathematical Olympiad 1989

Final (Fourth) Round

Grade 8

First Day

1. Find three different natural numbers in an arithmetic progression whose product is a perfect square.

2. Find the roots of the equation $x^2 + px + q = 0$, knowing that they are integral and that $p + q = 198$.

3. The numbers $1, 2, \dots, 12$ have been written around a circle in an arbitrary order. In one step it is allowed to exchange places of two adjacent numbers if their absolute difference is greater than 1. Prove that it is possible in a finite series of moves to rearrange the numbers in the natural order.

4. The diagonals of a convex quadrilateral $ABCD$ are mutually perpendicular. Perpendicular lines from the midpoints of sides AB and AD are dropped to their opposite sides CD and CB, respectively. Prove that these two lines and line AC have a common point.

Second Day

5. Show that number $4^{545} + 545^4$ is composite.

6. Give an example of a triangle which can be cut into (a) 12; (b) 5 congruent triangles.

7. The books of collected literary works in 20 volumes have been put on a bookshelf in an arbitrary order. The librarian wants to reorder them in increasing order, the 1-st to 20-th volume from left to right. In each move, the librarian takes a book not standing on its correct position and the book occupying the position of the first book, and exchanges their places. Prove that the number of such operations necessary to reorder the books does not depend on the librarian's moves.

8. A convex equilateral pentagon is placed within a circle. Each of its sides is extended to the intersection with the circle. The extensions of sides AB, BC, CD, DE, EA over B, C, D, E, A are painted in red, whereas their extensions over A, B, C, D, E are painted in blue. Prove that the total length of the red segments equals the total length of the blue segments.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

Grade 9

First Day

1. Prove that the sum $1 \cdot 2 \cdot 3 \cdots 2001 + 2002 \cdot 2003 \cdots 4002$ is divisible by 4003.

2. On each square of a 9×9 board sits a firefly. At a signal, each firefly moves to a diagonally adjacent square, whereby some squares may contain several fireflies while some other may be left unoccupied. Find the smallest possible number of unoccupied squares.

3. Let A be a fixed point on a ray of a given angle. Consider all circles touching this ray at A and intersecting the other ray at some points B and C. Prove that the incenters of all such triangles ABC lie on a line.

4. Four numbers $7956, 3923, 5857, 9725$ are written on the board one after another. The remainder of their sum divided by 10000 is written after them and the first number is erased; then this process is continued. Can four numbers equal to 1989 occur on the board at some moment?

Second Day

5. Seven hexagonal cells are assembled and colored white and blue as shown on picture 1 (the blue ones are shadowed). In each step it is allowed to choose one cell and change its color and the colors of the neighboring cells. Prove that, with finitely many recolorings,

   (a) one can obtain the coloring shown on picture 2;

   (b) one can not obtain the coloring shown on picture 3.

Pic.1 Pic.2 Pic.3

6. Prove that any numbers x, y, z from the interval $(0, 1)$ satisfy

$$x(1-y) + y(1-z) + z(1-x) < 1.$$

7. An arc BC is fixed on a circle. Consider all triangles ABC with A lying on the arc BC. For every such triangle, the line through the tangency points of the incircle with the sides AB and AC is drawn. Prove that all such lines are tangent to the same circle.

8. (a) Give an example of a triangle that can be cut into 13 congruent triangles.

   (b) Prove that for every n that can be written as a sum of two squares there exists a triangle that can be cut into n congruent triangles.

Grade 10

First Day

1. Given n, find natural numbers $a_1 < a_2 < \cdots < a_{2n+1}$ forming an arithmetic progression whose product is a perfect square.

2. The numbers $1, 2, \dots, n$ have been written around a circle in an arbitrary order. In one step it is allowed to exchange places of two adjacent numbers if their absolute difference is greater than 1. Prove that it is possible in a finite series of moves to rearrange the numbers in the natural order.

3. Points D and E are chosen on the legs AC and CB respectively of a right triangle ABC. Prove that the feet of the perpendiculars from C to DE, EA, AB, and BD lie on a circle.

4. Let $a, b, c \geq 0$ and $a + b + c \leq 3$. Prove the inequalities

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3}{2} \leq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

Second Day

5. Show that for any value of c the equation $x(x^2-1)(x^2-10) = c$ cannot have five integer solutions.

6. On the left-bottom square $a1$ of a chessboard there is a rook. In each move the rook moves to an adjacent square, horizontally or vertically. Can the rook travel around the chessboard, visiting one square exactly once, one square twice, etc, one square 64 times, and

   (a) return to the starting square $a1$?

   (b) end the travel on $a2$?

   (The initial position at $a1$ counts as one visit of that square.)

7. Three chords AA_1, BB_1, CC_1 of a circle (with the order on the circle A, B_1, C, A_1, B, C_1) pass through a common point K and form six angles of 60°. Prove that

$$KA + KB + KC = KA_1 + KB_1 + KC_1.$$

8. Can three non-intersecting regular tetrahedra with edge 1 be placed in a cube with edge 1? (The tetrahedra may touch each other.)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com