24-th All-Russian Mathematical Olympiad 1998

Fourth Round

Grade 8

First Day

1. Do there exist n-digit numbers M and N such that all digits of M are even, all digits of N are odd, all digits from 0 to 9 occur in M and N, and M divides N?
2. In a parallelogram $ABCD$ points M and N are the midpoints of BC and CD respectively. Can the rays AM and AN divide the angle BAD into three equal parts?
3. There are 52 cards in a deck, 13 of each suit. Vanya picks one by one card from the deck without putting it back. Every time he picks a card, he guesses its suit. Show that if Vanya always guesses a suit having no fewer remaining cards than any other suit, he will guess correctly at least 13 times.
4. A set of $n \geq 9$ points is given on the plane. For any nine of these points there are two circles on which all the nine points lie. Prove that all n points lie on two circles.

Second Day

5. Write the numbers 1 through 9 in the cells of a 3×3 table in such a way that the sum of numbers in any four cells forming a square (there are six squares) is the same.
6. A group of shepherds have 128 sheep in total. If one of them has at least half of the sheep, each other shepherd steals from him as many sheep as he already has. If two shepherds each have 64 sheep, someone takes a sheep from one of the two. Suppose seven rounds of theft occur. Prove that one shepherd ends up with all of the sheep.
7. In an acute-angled triangles ABC, O is the orthocenter and S_A, S_B, S_C the circles through O touching the sides BC, CA, AB respectively. Prove that the sum of the angle between the tangents to S_A at A, between the tangents to S_B at B, and between the tangents to S_C at C is equal to 180°.
8. In the city council elections, each voter, if he chooses to participate, votes for himself (if he is a candidate) and for all his friends. A pollster predicts the number of votes each candidate will receive. Show that the voters can conspire so that none of the pollster's numbers are correct.

Grade 9

First Day

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1. The three sides and the inradius of a triangle form an arithmetic progression. Find all such triangles.

2. Two circles intersect at P and Q. A line meets the first circle at A and C and the second at B and D in the order A, B, C, D. Prove that $\angle APB = \angle CQD$.

3. A 10-digit number is called *interesting* if it is a multiple of 11111 and all its digits are distinct. How many interesting numbers are there?

4. We have a 102×102 sheet of graph paper and a connected figure consisting of 101 squares. What is the smallest number of copies of the figure that can always be cut out of the square? A figure is connected if any two of its squares can be joined by a path along squares such that any two neighboring squares share a side.

Second Day

5. The roots of two quaratic trinomials are negative integers, and one root is common. Can the values of the trinomials at some positive integer point be 19 and 98?

6. On the outermost squares of a 1×101 board there are two pieces: the first player's piece on the left and the second player's on the right. A player in turn moves his piece for 1,2,3 or 4 squares towards the opponent's end of the board. It is allowed to jump over the opponent's piece, but not to step on the same square with it. The player who first reaches the opposite end of the board wins. Who wins if both players are intelligent enough?

7. A pool-table in the shape of a regular 1998-gon $A_1 \dots A_{1998}$ is given. A ball isn hit from the midpoint of A_1A_2 and, after reflecting from the sides $A_2A_3, A_3A_4, \dots, A_{1998}A_1$ respectively (so that the angle of incidence equals the angle of reflection), it returns to the starting point. Prove that the trajectory of the ball is a regular 1998-gon.

8. The endpoints of a compass are at two lattice points of an infinite unit square grid. It is allowed to rotate the compass around one of its endpoints, not varying its radius, and thus move the other endpoint to another lattice point. Can the endpoints of the compass change places after several such steps?

Grade 10

First Day

1. Let $f(x) = x^2 + ax + b\cos x$. Find all values of a and b for which the sets of the solutions of the equations $f(x) = 0$ and $f(f(x)) = 0$ are equal and nonempty.

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2. In an acute-angled triangle ABC the circle through B, C and the circumcenter O meets the lines AB and AC again at D and E respectively. Let OK be the diameter of this circle. Show that $ADKE$ is a parallelogram.

3. Show that, given any finite set of points in the plane, one can remove one point and divide the remaining set into two subsets each with smaller diameter than the original set. (The diameter of a finite set of points is the maximum distance between two points in the set.)

4. In 1999 memory locations of a computer are stored numbers $1, 2, 2^2, \dots, 2^{1998}$. Two programmers take turns subtracting 1 from each of five different locations. If any location acquires a negative number, the computer breaks and the guilty programmer pays for the repairs. Which programmer can ensure himself not to pay, and how?

Second Day

5. Solve the equation $\left\{(x+1)^3\right\} = x^3$, where $\{z\} = z - [z]$ is the fractional part of z.

6. In the pentagon $A_1A_2A_3A_4A_5$ are drawn the bisectors $l_1, \dots, l_5$ of the angles $A_1, \dots, A_5$, respectively. Bisectors l_1 and l_2 meet at B_1, l_2 and l_3 meet at B_2, etc, l_5 and l_1 meet at B_5. Can the pentagon $B_1B_2B_3B_4B_5$ be convex?

7. A cube of edge n is divided into unit cubes by cuts between adjacent unit cubes. What is the smallest number of cuts that can be undone so that from each cube one can reach the surface of the large cube without passing through a cut?

8. A number from 1 to 144 is chosen. It is allowed to pick a subset of the numbers $1, \dots, 144$ and ask whether the chosen number is in the set. An answer "yes" is charged 2 rubles, and an answer "no" 1 ruble. What smallest amount of money is always sufficient to determine the chosen number?

Grade 11

First Day

1. Two decks with 36 cards each are given. The first deck is shuffled and put on the other one. Then for each card of the first deck the number of cards between that card and the corresponding card of the other deck is taken. What is the sum of the 36 obtained numbers?

2. A circle $\mathscr{S}$ with center O intersects a circle $\mathscr{S}'$ at points A and B. Point C is taken on the arc of $\mathscr{S}$ inside $\mathscr{S}'$. Lines AC and BC meet $\mathscr{S}'$ again in points E and D respectively. Prove that DE and OC are perpendicular.

3. *Problem 3 for Grade 10.*

4. In $n-1$ squares of an $n \times n$ board ($n > 100$) are written 1's, while the other squares are filled with 0's. The following operation is allowed: Choose a square, decrease the number in it by 1 and increase all other numbers in the same row or column by 1. Is it possible to obtain a board with all numbers equal after finitely many such operations?

Second Day

5. An integer is written on the blackboard. Its last digit is repeatedly erased, multipled by 5 and then added to the rest of the number. Starting with number 7^{1998}, shall we ever obtain 1998^7?

6. On an infinite chessboard is drawn a polygon with sides along the grid lines. Each unit segment on its perimeter is colored black or white according to whether it touches a black or white square inside the polygon. Let A and B be the numbers of black and white segments on the perimeter and a and b be the numbers of black and white squares inside the polygon. Prove that $A - B = 4(a - b)$.

7. Two regular tetrahedra of edge $\sqrt{2}$ are the images of each other under a planar reflection. Let Φ be the set of midpoints of all segments joing points on the two tetrahedra. Find the volume of the figure Φ.

8. In a sequence $(a_n)_{n\in\mathbb{N}}$ of natural numbers every natural number occurs and for any distinct natural numbers m and n

$$\frac{1}{1998} < \frac{|a_n - a_m|}{|n-m|} < 1998.$$

Prove that $|a_n - n| < 2,000,000$ for all natural n.

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