21-th All-Russian Mathematical Olympiad 1995

Fourth Round

Grade 9

First Day

1. If x and y are positive numbers, prove the inequality
$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\leq\frac{1}{xy}.$$

2. Is it possible to write 1995 different natural numbers around a circle so that the ratio of every two adjacent numbers is a prime number?

3. Two circles with radii R and r intersect at C and D and are tangent to a line l at A and B. Prove that the circumradius of triangle ABC does not depend on the length of segment AB.

4. Every side and diagonal of a regular 12-gon is colored in one of 12 given colors. Can this be done in such a way that, for every three colors, there exist three vertices which are connected to each other by segments of these three colors?

Second Day

5. Find all prime numbers p for which number p^2+11 has exactly six different divisors (counting 1 and itself).

6. Circles $\mathscr{S}_1$ and $\mathscr{S}_2$ with centers O_1 and O_2 respectively intersect at A and B. The circle passing through O_1, O_2, and A intersects $\mathscr{S}_1, \mathscr{S}_2$ and line AB again at D, E, and C, respectively. Show that $CD = CB = CE$.

7. A regular hexagon of side 5 is cut into unit equilateral triangles by lines parallel to the sides of the hexagon. We call the vertices of these triangles *knots*. If more than half of all knots are marked, show that there exist five marked knots that lie on a circle.

8. Can the numbers $1, 2, \dots, 121$ be written in the cells of an 11×11 board in such a way that any two consecutive numbers are in adjacent cells (sharing a side), and all perfect squares are in the same column?

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Grade 10

First Day

1. Given function $f(x) = \dfrac{1}{\sqrt[3]{1-x^3}}$, find $\underbrace{f(\cdots f(f(19))\cdots)}_{95}$.

2. Natural numbers m and n satisfy $\gcd(m,n)+\operatorname{lcm}(m,n) = m+n$. Prove that one of numbers m,n divides the other.

3. In an acute-angled triangle ABC, the circle $\mathscr{S}$ with the altitude BK as the diameter intersects AB at E and BC at F. Prove that the tangents to $\mathscr{S}$ at E and F meet on the median from B.

4. There are several equal (possibly overlapping) square-shaped napkins on a rectangular table, with sides parallel to the sides of the table. Prove that it is possible to nail some of them to the table in such a way that every napkin is nailed exactly once.

Second Day

5. Consider all quadratic functions $f(x) = ax^2+bx+c$ with $a < b$ and $f(x) \geq 0$ for all x. What is the smallest possible value of the expression $\frac{a+b+c}{b-a}$?

6. Let $ABCD$ be a quadrilateral with $AB = AD$ and $\angle ABC = \angle ADC = 90^\circ$. Points F and E are chosen on the respective sides BC and CD so that $DF \perp AE$. Prove that $AF \perp BE$.

7. N^3 unit cubes are made into beads by drilling a hole through them along a diagonal, put on a string and binded. Thus the cubes can move freely in space as long as the vertices of two neighboring cubes (including the first and last one) are touching. For which N is it possible to build a cube of edge N using these cubes?

8. The streets of the city of Duzhinsk are simple polygonal lines not intersecting each other in internal points. Each street connects two crossings and is colored in one of three colors: white, red, or blue. At each crossing exactly three streets meet, one of each color. A crossing is called *positive* if the streets meeting at it are white, blue and red in counterclockwise direction, and *negative* otherwise. Prove that the difference between the numbers of positive and negative crossings is a multiple of 4.

Grade 11

First Day

1. *Problem 1 for Grade 10.*

2. A planar section of a parallelepiped is a regular hexagon. Show that this parallelepiped is a cube.

3. *Problem 4 for Grade 9.*

4. There are finitely many congruent, parallelly positioned squares on a plane such that among any $k+1$ squares some two intersect. Show that the squares can be divided into at most $2k-1$ nonempty groups such that all squares in the same group have a common point.

Second Day

5. Angles α, β, γ satisfy the inequality $\sin\alpha + \sin\beta + \sin\gamma \geq 2$. Prove that $\cos\alpha + \cos\beta + \cos\gamma \leq \sqrt{5}$.

6. A sequence $a_0, a_1, a_2, \dots$ satisfies $a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$ for all nonnegative integers m, n with $m \geq n$. Given that $a_1 = 1$, find a_{1995}.

7. Circles $\mathscr{S}_1$ and $\mathscr{S}_2$ with centers O_1 and O_2 respectively intersect at A and B. Ray O_1B meets $\mathscr{S}_2$ again at F, and ray O_2B meets $\mathscr{S}_1$ again at E. The line through B parallel to EF intersects $\mathscr{S}_1$ and $\mathscr{S}_2$ again at M and N, respectively. Prove that $MN = AE + AF$.

8. *Problem 8 for Grade 10.*

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