# 18-th All-Russian Mathematical Olympiad 1992

## Final (Fourth) Round – March 22–29

### Grade 9

*First Day*

1. Solve the system of equations
$$x^3 - 5\frac{y^2}{x} = \frac{6}{y}, \quad y^3 - 5\frac{x^2}{y} = \frac{6}{x}.$$

2. Two players alternately put checkers on the cells of a $99 \times 99$ board. A player can put a checker on some cell if all neighboring cells are free or there is a checker put by his opponent on one of the neighboring cells (two cells are neighboring if they have a common side). The player who cannot make a legal move loses. Who has a winning strategy?

3. A circle inscribed in a deltoid $ABCD$ with $AB = BC$ and $AD = DC$ touches the sides $AB, BC, AD$ at $K, M, N$, respectively. The diagonal $AC$ intersects $MN$ at $P$. Prove that points $A, K, P, N$ lie on a circle.

4. A city has the shape of a square of side $n-1$ divided by roads into $(n-1)^2$ unit squares. Some two-way bus routes are to be established in such a way that every route has at most one turn and it is possible to travel between any two crossroads with at most one transfer. How many bus routes at least are necessary to achieve this?

*Second Day*

5. From a chessboard, (a) cell b2, (b) cells b2 and g2 are cut off. Can a piece, starting at cell C2 and only moving from one cell to a neighboring (sharing a side) cell, move around this chessboard so as to visit every cell exactly once?

6. Solve in prime numbers the equation $2^{x+1} + y^2 = z^2$.

7. Points $B_1$ and $C_1$ are taken on side $BC$ of a triangle $ABC$ such that $BB_1 = CC_1$ and $\angle BAB_1 = \angle CAC_1$. Prove that the triangle $ABC$ is isosceles.

8. Cities of a certain country are connected by air routes (each connecting two cities) served by $2k+1$ companies, where the first company serves one route, the second serves two, etc. According to a law in this country, a company cannot serve more than one route from the same city. Some day, the companies agreed to redistribute the routes between themselves so that each company serves the same number of routes. Show that this can be done without violating the law.



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## Grade 10

*First Day*

1. In a $7 \times 7$ board 19 cells are colored. We say that a row or column is colored if it contains at least four colored cells. How many colored rows and columns at most can there be in the board?

2. If $a, b, c$ are positive parameters, solve the system of equations

$$\frac{a}{x} - \frac{b}{z} = c - zx, \quad \frac{b}{y} - \frac{c}{x} = a - xy, \quad \frac{c}{z} - \frac{a}{y} = b - yz.$$

3. (a) Infinitely many reflectors are placed at some integer points in the positive quadrant of the coordinate plane. Each reflector illuminates the angle whose rays are parallel to the coordinate axes and point in the same direction. Show that it is possible to turn off all the reflectors except finitely many, so that all illuminated points remain illuminated.

   (b) The same problem with reflectors illuminating trihedral angles, placed at some integer points in the positive octant of the coordinate space.

4. In a pentagon $ABCDE$ with $BC \parallel AD$ and $BD \parallel AE$, $M$ and $N$ are the midpoints of $CD$ and $DE$ respectively, and $O$ the intersection of $BN$ and $AM$. Prove that the areas of the quadrilateral $MDNO$ and the triangle $ABO$ are equal.

*Second Day*

5. Solve the equation $x + \dfrac{92}{x} = [x] + \dfrac{92}{[x]}$.

6. Let $D$ be a point on side $BC$ of a triangle $ABC$, and let $O, O_1, O_2$ be the circumcenters of triangles $ABC, ABD$, and $ADC$. Show that the points $O, O_1, O_2, A$ lie on a circle.

7. Consider all possible sets of $n$ weights whose masses in grammes are distinct integers not exceeding 21. What is the smallest $n$ such that in every such set of weights there are two pairs whose total masses are equal?

8. *Problem 8 for Grade 9.*

## Grade 11

*First Day*

1. *Problem 1 for Grade 10.*

2. *Problem 2 for Grade 10.*

3. There are several cities in a country. Some of the cities are connected by one-way airlines. It is known that there exists a city from which not every city can be reached. Prove that one can select a group of cities such that none of the cities in the group can be reached from any city not in the group.

4. A finite set of points in the plane, no three of which are collinear, has the property that for every three points $A, B, C$ from the set the orthocenter of the triangle $ABC$ is also in the set. Find all such sets.

*Second Day*

5. The sequence $(a_n)$ is defined by $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{[a_n]}$ for $n \geq 1$. For which values of $n$ does the inequality $a_n > 20$ hold?

6. Does there exist a set $M$ of lines in space with the following properties:

   (i) Every point in space lies on exactly two lines from $M$;

   (ii) Every two points can be connected to each other by a polygonal line with segments going along lines from $M$?

7. In a triangle $ABC$, $D$ is a point on side $BC$ different from its midpoint, and $O_1, O_2$ the circumcenters of triangles $ABD$ and $ADC$. Prove that the perpendicular bisector of the median $AK$ of triangle $ABC$ bisects the segment $O_1O_2$.

8. There are $N$ stones in a pile. Two players alternately take stones from the pile. A player in turn takes a number of stones that divides the number of stones taken by his opponent in his last move. The first player in his first move can take an arbitrary number of stones, but at least one and not all $N$. The player who picks up the last stone wins the game. What is the smallest $N > 1992$ for which the second player has a winning strategy?





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